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Gravitating Self-dual Chern-Simons Solitons

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Abstract

Self-dual solitons of Chern-Simons Higgs theory are examined in curved spacetime. We derive duality transformation of the Einstein Chern-Simons Higgs theory within path integral formalism and study various aspects of dual formulation including derivation of Bogomolnyi type bound. We find all possible rotationally-symmetric soliton configurations carrying magnetic flux and angular momentum when underlying spatial manifolds of these objects comprise a cone, a cylinder, and a two sphere.

1 Introduction

Since the Chern-Simons gauge field made appearance in physics literature as a mass term in three dimensional gauge theory [1], it has attracted the attention in various ways, e.g., anyon and fractional statistics [2], a relation to knot theory [3], and conformal field theories in two dimensions [4]. Chern-Simons gauge theories coupled to scalar matter field also support the soliton solutions carrying fractional angular momentum [5, 6]. An example which provides static multi-soliton solutions is relativistic Chern-Simons Higgs model of a specific ϕ^6 scalar potential, and it admits both topological and nontopological solitons [7, 8, 9]. For the vortices in Abelian Higgs model with critical coupling, it was proved in Ref. [10] that the Bogomolnyi type bound is saturated after the inclusion of Einstein gravity. The progress to this direction has also been achieved in Chern-Simons Higgs model coupled to background gravity [11] and Einstein gravity [12]. The solitons of Abelian Higgs model are distinguished from those of Chern-Simons gauge theories by the fact that whether they are spinning objects or not. This angular momentum affects the derivation of Bogomolnyi bound in curved space: the Bogomolnyi limit of Nielsen-Olesen vortices is obtained under the static metric and the same ϕ^4 scalar potential as in flat space case, however that of the self-dual Chern-Simons solitons is saturated under the general stationary metric and thereby introduce a ϕ^8 scalar potential of negative coefficient in addition to the ϕ^6 potential of flat spacetime [12]. Furthermore, the scalar potential in the Bogomolnyi limit of the Einstein Chern-Simons Higgs model includes, at least, one parameter, even if we use the condition that the obtained first-order equations reproduce original second-order Euler-Lagrange equations and their solutions carry finite magnetic flux (or equivalently charge) and angular momentum.

In this paper, for the static configurations of Chern-Simons Higgs model, we shall derive conventional Bogomolnyi type bound of the total energy defined by Euler number in terms of magnetic flux, the topological charge of the system. Examining the obtained Bogomolnyi equations by use of both analytic and numerical methods, we shall show that there exist only two types of smooth solutions despite the complicated ϕ^8 potential unbounded below. They are topological vortices and nontopological solitons which coincide exactly with the solitonic spectra in flat spacetime. First, when boundary value of the scalar field has a Higgs vacuum, self-dual vortex solutions are supported and the underlying manifolds of these

solutions are asymptotic cone and cylinder. Second, when boundary value of the scalar field has a symmetric local minimum, self-dual nontopological solitons are produced and the corresponding spatial manifold constitutes two sphere, asymptotic cylinder in addition to asymptotic cone. It is an intriguing point that the scalar potential vanishes at the boundary value of both solitons in open spatial manifold, which implies zero cosmological constant at asymptotic open space. Examining asymptotic behaviors of solutions for multi-solitons, we conclude that there exists an upperbound for the vorticity to form a vortex and the obtained solutions are decaying fast as the radial distance increases.

A way to envisage the role of topological excitations in the path integral formalism of Abelian gauge theories is to reformulate the given theory through the dual formulation. It has been studied for the lattice version of the Abelian Higgs model [13], and for the continuum version of the Chern-Simons Higgs model [14]. In this paper, we shall rewrite the Chern-Simons Higgs model coupled to gravity in terms of dual gauge field. In the dual transformed theory, we obtain the explicit forms of the nonpolynomial interaction between the dual gauge and Higgs fields, and the topological interaction between the dual gauge field and the topological sector of the scalar phase. Furthermore, we demonstrate a role of those interactions in order to produce nonperturbative excitations and their mutual interactions. For gravitational field, the path integral measure of the duality transformed theory possesses a Jacobian relative to that of original theory.

In the next section, we will introduce the Chern-Simons Higgs model in curved spacetime and rederive the Bogomolnyi type bound. In section 3, we rewrite our model through the duality transformation and discuss physics of the topological excitations comparing with the original theory. In section 4, we examine the Bogomolnyi equations and obtain all possible regular rotationally-symmetric solutions. We show that the decaying property of solutions of the Bogomolnyi equations and nonexistence of solutions with prescribed vortices in section 5. Conclusions with some comments about our results are presented in section 6.

2 Self-dual Chern-Simons Solitons in Curved Space

In this section, we recapitulate the derivation of the Bogomolnyi bound for Chern-Simons solitons in curved space. In order to obtain such Bogomolnyi-type bound, we consider the

energy defined by the spatial integration of two dimensional scalar curvature instead of the matter action for static configurations as has been done in Ref. [12].

The action for an Abelian Chern-Simons gauge field theory with Higgs mechanism is written in curved spacetime

$$S = \int d^3x \sqrt{g} \left[-\frac{1}{16\pi G} R + \frac{\kappa}{2} \frac{\epsilon^{\mu\nu\rho}}{\sqrt{g}} A_\mu \partial_\nu A_\rho + \frac{1}{2} g^{\mu\nu} \overline{D}_\mu \phi D_\nu \phi - V(|\phi|) \right], \quad (2.1)$$

where $\phi = |\phi|e^{i\Omega}$, gauge covariant derivative is $D_\mu = \partial_\mu - ieA_\mu$, and scalar potential $V(|\phi|)$ will be fixed to give a suitable Bogomolnyi type bound. Dynamics of gravitation is governed by 2+1 dimensional Einstein gravity and the metric takes the stationary form, which is compatible with static spinning objects

$$ds^2 = N^2(dt + K_i dx^i)^2 - \gamma_{ij} dx^i dx^j. \quad (2.2)$$

In the above equation the components of the metric, N, K_i , and γ_{ij} , ($i, j = 1, 2$), do not depend on time variable.

Variation of the action with respect to the fields leads to the equations of motion

$$\frac{1}{\sqrt{g}} D_\mu (\sqrt{g} g^{\mu\nu} D_\nu \phi) = -\frac{\phi}{|\phi|} \frac{dV}{d|\phi|}, \quad (2.3)$$

$$\frac{\kappa}{2} \frac{\epsilon^{\mu\nu\rho}}{\sqrt{g}} F_{\nu\rho} = e j^\mu, \quad (2.4)$$

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}. \quad (2.5)$$

In Eq. (2.5) $T_{\mu\nu}$ denotes symmetric energy-momentum tensor

$$T_{\mu\nu} = \frac{1}{2} \left(\overline{D}_\mu \phi D_\nu \phi + \overline{D}_\nu \phi D_\mu \phi \right) - g_{\mu\nu} \left(\frac{1}{2} g^{\rho\sigma} \overline{D}_\rho \phi D_\sigma \phi - V \right). \quad (2.6)$$

This theory possesses a local U(1) gauge symmetry so that it admits a conserved U(1) charge

$$Q = \int d^2x \sqrt{\gamma} j^0, \quad (2.7)$$

where $\gamma = \det \gamma_{ij}$ and j^0 is time-component of the U(1) current

$$j^\mu = -\frac{i}{2} \left(\bar{\phi} D^\mu \phi - \overline{D^\mu \phi} \phi \right). \quad (2.8)$$

We are interested in the static soliton-like excitations satisfying the classical equations of motion (2.3)-(2.5), and particularly the self-dual solitons supported by first-order Bogomolnyi

equations. Though it is in general difficult to have the expression for total energy of the system, we take it as an Euler invariant [19, 20]. From time-component of the Einstein equations (2.5), we have the energy expression in terms of matter fields

$$\begin{aligned} E &= \frac{1}{16\pi G} \int d^2x \sqrt{\gamma} {}^2R \\ &= \int d^2x \sqrt{\gamma} \left\{ \frac{e^2}{2N^2} A_0^2 |\phi|^2 + \frac{1}{2} \gamma^{ij} \tilde{D}_i \phi \tilde{D}_j \phi + V(|\phi|) - \frac{3}{32\pi G} N^2 K^2 \right\}, \end{aligned} \quad (2.9)$$

where 2R is two dimensional scalar curvature, $\tilde{D}_i \phi = (\partial_i - ieA_i + ieK_i A_0) \phi$, $K = \frac{\epsilon^{ij}}{\sqrt{\gamma}} \partial_i K_j$, and ϵ^{ij} is two dimensional Levi-Civita tensor density of $\epsilon^{12} = \epsilon_{12} = 1$. Suppose that the potential V is made of both G -independent U and G -dependent W such as $V = \frac{1}{2} U^2 + W$. We can rearrange the terms in the integrand of right-hand side of Eq. (2.9) under an assumption $N(x^i) = 1$ as follows

$$\begin{aligned} E &= \int d^2x \sqrt{\gamma} \left\{ \frac{e^2}{2} |\phi|^2 \left(A_0 \pm \frac{U}{e|\phi|} \right)^2 \right. \\ &\quad + \frac{1}{4} \gamma^{ij} (\tilde{D}_i \phi \mp i\sqrt{\gamma} \epsilon_{ik} \gamma^{kl} \tilde{D}_l \phi) (\tilde{D}_j \phi \mp i\sqrt{\gamma} \epsilon_{jm} \gamma^{mn} \tilde{D}_n \phi) \\ &\quad \mp \frac{e}{2} (|\phi|^2 - v^2) \left[\frac{\epsilon^{ij}}{\sqrt{\gamma}} \partial_i (A_j - K_j A_0) + K A_0 + \frac{e^2}{\kappa} A_0 |\phi|^2 \right] \\ &\quad \mp e A_0 |\phi| \left[U - \frac{e^2}{2\kappa} |\phi| (|\phi|^2 - v^2) \right] \\ &\quad \left. + \left[W + \frac{ev^2}{2} K \left(\pm \frac{A_0}{v^2} (|\phi|^2 - v^2) + \frac{ev^2}{2\kappa} C \right) - \frac{3}{32\pi G} K^2 \right] \right\} \\ &\quad \pm \frac{ev^2}{2} \Phi \\ &\quad + \int d^2x \partial_i \left[\epsilon^{ij} \left(\pm \frac{ev^2}{2} K_j (A_0 \mp \frac{ev^2}{2\kappa} C) \pm \frac{i}{4} (\bar{\phi} \tilde{D}_j \phi - \phi \overline{\tilde{D}_j \phi}) \right) \right], \end{aligned} \quad (2.10)$$

where Φ is the magnetic flux defined by

$$\Phi = -\frac{1}{2} \int d^2x \epsilon^{ij} F_{ij}, \quad (2.11)$$

and C a constant introduced so as to make the surface term on the last line vanish. Since the spinning objects specified by the nonzero angular momentum J given as

$$J = \frac{1}{8\pi G} \oint_{|\vec{x}| \rightarrow \infty} dx^i K_i \quad (2.12)$$

are of our interest, the requirement that the surface term in the last line of Eq. (2.10) has no contribution to the energy relates C to the value of scalar magnitude at spatial infinity $\phi_\infty = \lim_{|\vec{x}| \rightarrow \infty} \frac{|\phi|}{v}$ such as

$$C = 1 - \phi_\infty^2. \quad (2.13)$$

The offdiagonal metric components K in Eq. (2.10) are replaced by matter fields from $0i$ -components of the Einstein equations (2.5)

$$K = -8\pi G\kappa \frac{(A_0^2 - \frac{e^2 v^4}{4\kappa^2} D)}{N^3}, \quad (2.14)$$

where D is an integration constant.

Now let us choose a scalar potential to achieve a Bogomolnyi-type bound. Specifically, U is chosen to make the square bracket on the fourth line of Eq. (2.10) zero, and thereby the condition, that the first line of Eq. (2.10) should vanish, forces the auxiliary field A_0 to be

$$A_0 = \mp \frac{U}{e|\phi|} = \mp \frac{e}{2\kappa} (|\phi|^2 - v^2). \quad (2.15)$$

The third line of Eq. (2.10) does not contribute to the energy because of the Gauss' law given by time-component of the Chern-Simons equations (2.4). We read G -dependent terms of the scalar potential W as an eighth-order term from the fifth line of Eq. (2.10). Hence, for the configurations to satisfy the first-order equation

$$\tilde{D}_i \phi \mp i\sqrt{\gamma} \epsilon^{ij} \gamma^{jk} \tilde{D}_k \phi = 0, \quad (2.16)$$

the energy is proportional to the magnetic flux and the Bogomolnyi-type bound is saturated if such solitons carry magnetic flux. The Gauss' law says that these flux-carrying solitons are charged objects and nontopological solitons can also be produced when the meson mass in the symmetric phase of the theory is not larger than $e^2 v^2 / 2\kappa$. The remaining equations are ij -components of the Einstein equations (2.5) summarized as follows

$$\begin{aligned} & \frac{1}{8\pi G N} (\gamma^{ij} \nabla^2 - \nabla^i \nabla^j) N \\ &= \frac{e^2}{2} \gamma^{ij} |\phi|^2 \left(\frac{A_0}{N} - \frac{U}{e|\phi|} \right) \left(\frac{A_0}{N} + \frac{U}{e|\phi|} \right) - \gamma^{ij} \left(W + \frac{1}{32\pi G} N^2 K \right) \\ &+ \frac{1}{8} \left\{ \left[(\tilde{D}^i \phi \mp i \frac{\epsilon^{ik}}{\sqrt{\gamma}} \gamma_{kl} \tilde{D}^l \phi) (\tilde{D}^j \phi \pm i \frac{\epsilon^{jm}}{\sqrt{\gamma}} \gamma_{mn} \tilde{D}^n \phi) \right. \right. \end{aligned}$$

$$\begin{aligned}
& + \overline{(\tilde{D}^j \phi \pm i \frac{\epsilon^{jk}}{\sqrt{\gamma}} \gamma_{kl} \tilde{D}^l \phi)(\tilde{D}^i \phi \mp i \frac{\epsilon^{im}}{\sqrt{\gamma}} \gamma_{mn} \tilde{D}^n \phi)} \\
& + \left[\overline{(\tilde{D}^i \phi \pm i \frac{\epsilon^{ik}}{\sqrt{\gamma}} \gamma_{kl} \tilde{D}^l \phi)(\tilde{D}^j \mp i \frac{\epsilon^{jm}}{\sqrt{\gamma}} \gamma_{mn} \tilde{D}^n \phi)} \right. \\
& \left. + \overline{(\tilde{D}^j \phi \mp i \frac{\epsilon^{jk}}{\sqrt{\gamma}} \gamma_{kl} \tilde{D}^l \phi)(\tilde{D}^i \phi \pm i \frac{\epsilon^{im}}{\sqrt{\gamma}} \gamma_{mn} \tilde{D}^n \phi)} \right] \Big\}. \tag{2.17}
\end{aligned}$$

Note that any self-dual soliton solution of the Bogomolnyi equations with $N = 1$ automatically satisfies the above Einstein equations.

From now on we fix the conformal gauge for γ_{ij}

$$\gamma_{ij} = \delta_{ij} b(x^i), \tag{2.18}$$

and the Coulomb gauge for K^i , $\nabla_i K^i = 0$, which makes K^i be expressed as follows

$$K^i = -\frac{\kappa}{e^2 v^2} \frac{\epsilon^{ij}}{\sqrt{\gamma}} \partial_j \ln \psi. \tag{2.19}$$

Substituting Eqs. (2.18)-(2.19) into 00-component of the Einstein equations (2.5) and replacing the gauge field to the scalar field by use of the Bogomolnyi equation (2.16)

$$A_i - K_i A_0 = \frac{1}{e} (\partial_i \Omega \mp \epsilon^{ij} \partial_j \ln |\phi|), \tag{2.20}$$

we obtain an expression of the metric function b as

$$b = e^{h(\tilde{z}) + \bar{h}(\bar{\tilde{z}})} \left(\frac{f^2 e^{-(f^2-1)} \psi^{1-\phi_\infty^2}}{\prod_{p=1}^n |\tilde{z} - \tilde{z}_p|^2} \right)^{\tilde{G}}. \tag{2.21}$$

In the above $h(\tilde{z})$ ($\bar{h}(\bar{\tilde{z}})$) is a holomorphic (an anti-holomorphic) function and the variables with tilde are dimensionless quantities

$$\tilde{z} = \tilde{x}^1 + i\tilde{x}^2 = \frac{e^2 v^2}{|\kappa|} (x^1 + ix^2), \quad f = \frac{|\phi|}{v}, \quad \tilde{G} = 4\pi G v^2. \tag{2.22}$$

If we eliminate the gauge field and the metric b by use of Eqs. (2.16)-(2.21), we have a Bogomolnyi equation from the Gauss' law which is the time-component of the Chern-Simons equations (2.4) for the gauge field

$$\begin{aligned}
\tilde{\partial}^2 \ln f^2 &= e^{h(\tilde{z}) + \bar{h}(\bar{\tilde{z}})} \left(\frac{f^2 e^{-(f^2-1)} \psi^{1-\phi_\infty^2}}{\prod_{p=1}^n |\tilde{z} - \tilde{z}_p|^2} \right)^{\tilde{G}} (f^2 - 1) \\
&\times \left[f^2 - \frac{\tilde{G}}{2} (f^4 - 2f^2 + \phi_\infty^2) \right] \mp 2\epsilon^{ij} \tilde{\partial}_i \tilde{\partial}_j \Omega, \tag{2.23}
\end{aligned}$$

and the equation for ψ from Eq. (2.14)

$$\tilde{\partial}^2 \ln \psi = -\frac{\tilde{G}}{2} e^{h(\tilde{z}) + \bar{h}(\tilde{z})} \left(\frac{f^2 e^{-(f^2-1)} \psi^{1-\phi_\infty^2}}{\prod_{p=1}^n |\tilde{z} - \tilde{z}_p|^2} \right)^{\tilde{G}} (f^4 - 2f^2 + \phi_\infty^2), \quad (2.24)$$

where $\tilde{\partial}^2$ is flat-space Laplacian, i.e., we will use $\tilde{\partial}^2 \equiv \tilde{\partial}_i \tilde{\partial}_i = \Delta$.

The undetermined constant D in Eq. (2.14) is fixed by the requirement that the above equations (2.23)-(2.24) reproduce the scalar equation (2.3), $D = 1 - \phi_\infty^2$. In synthesis, the scalar potential becomes

$$V(|\phi|) = \frac{e^4}{8\kappa^2} \left[|\phi|^2 (|\phi|^2 - v^2)^2 - \pi G (|\phi|^4 - 2v^2 |\phi|^2 + v^4 \phi_\infty^2) \right]. \quad (2.25)$$

Though the Bogomolnyi limit of a theory selects usually a unique scalar potential in curved spacetime [10], shape of the scalar potential (2.25) depends on the boundary value of the scalar field ϕ_∞ (see also Fig. 1). This potential includes G -dependent eighth-order potential, and it is not bounded below since the coefficient of the eighth-order term is negative for positive Newton's constant G as shown in Fig. 1. However the energy in order to support flux-carrying solitons of the Bogomolnyi equations is positive definite. Inserting the metric (2.21) into the Euler characteristic (2.9), we have

$$\begin{aligned} & \frac{1}{4\pi G v^2} \int d^2x \sqrt{\gamma} {}^2R \\ &= \frac{1}{4\pi G v^2} \left\{ \int d^2x \partial^2 \ln \prod_{p=1}^n |z - z_p|^2 - \int d^2x \partial^2 \ln |\phi|^2 + \int d^2x \partial^2 \frac{|\phi|^2}{v^2} \right. \\ & \quad \left. - (1 - \phi_\infty^2) \int d^2x \partial^2 \ln \psi \right\}. \end{aligned} \quad (2.26)$$

From the above expression there can exist contributions from the second and fourth terms besides the first term for $n \neq 0$ if the Bogomolnyi equations (2.23)-(2.24) contain the finite energy solution which behaves as $|\phi| \sim |x^i|^{-\varepsilon}$ ($\varepsilon > 0$) for large $|\vec{x}|$, and $\psi \sim |x^i|^{-2\alpha}$ (α is an arbitrary number) when $\phi_\infty \neq 1$. We shall present the detailed analysis for the existence of such solutions in section 4.

Looking at shapes of the scalar potential in Fig. 1, one may raise an intriguing question in connection with local supersymmetry in 2+1 dimensions. According to Ref. [22], a well-known argument in 2+1 dimensions is that local supersymmetry can ensure the vanishing of the cosmological constant without requiring the equality of boson and fermion masses.

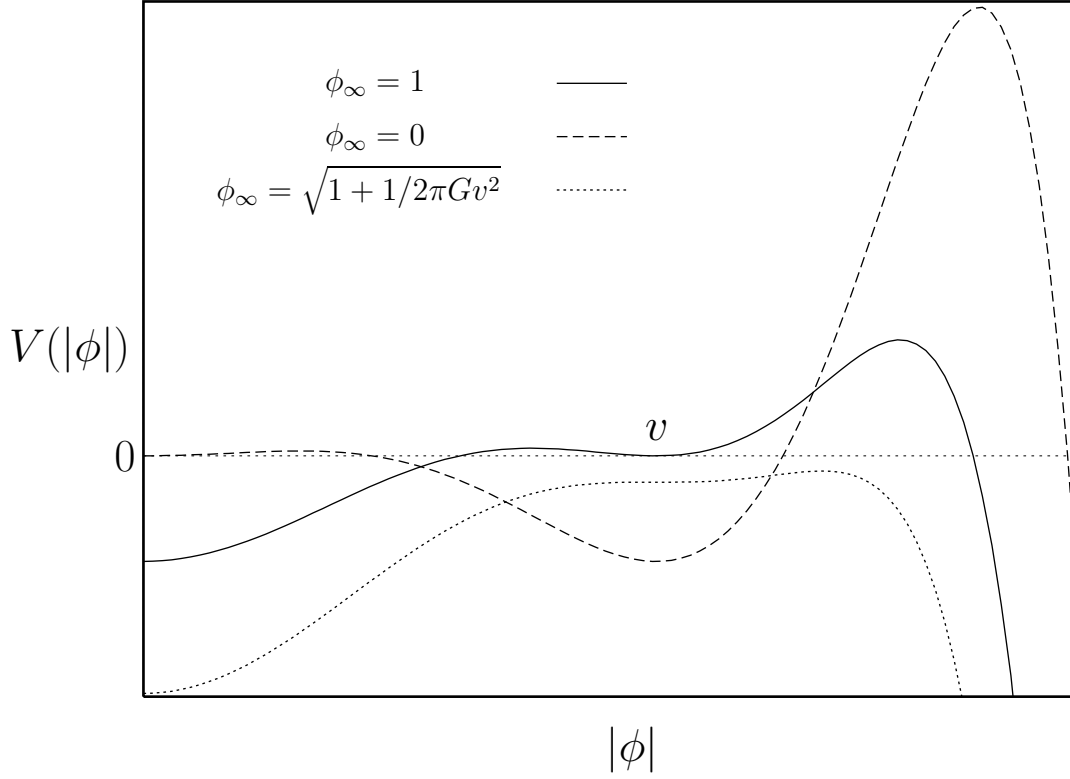


Figure 1: Schematic shapes of the scalar potential $V(|\phi|)$ for $e^4/8\kappa^2 = 1$ and $\pi G v^2 = 1$. The solid, dashed, and dotted lines correspond to $\phi_\infty = 1$, 0 , and $\sqrt{1 + 1/2\pi G v^2}$.

Suppose that the above theory is bosonic part of a presumed supergravity theory as has been done for $N = 2$ supersymmetric theory of self-dual Chern-Simons Higgs model in flat spacetime [21]. It is questionable that whether or not the supergravity version of our model satisfies the above criterion because of the following reasons. Although we let the cosmological constant vanish from the beginning, the scalar potential (2.25) is unbounded below and has negative local minima. Specifically, degeneracy between the symmetric vacuum, $|\phi| = 0$, and the broken vacua, $|\phi| = v$, disappears because of G -dependent terms, and this phenomenon is guaranteed by global supersymmetry [21] (see Fig. 1). In more detail, the value of the scalar potential for the boundary values of soliton configurations are calculated as follows;

$$V(|\phi| = v\phi_\infty) = \frac{\pi G e^4 v^8}{8 \kappa^2} \phi_\infty^2 (\phi_\infty^2 - 1)^2 \left(\phi_\infty^2 - \frac{1}{\pi G v^2} \right). \quad (2.27)$$

If the boundary value of the scalar field is different from 0, 1, $1/\sqrt{\pi G v^2}$, then $V(\phi_\infty) \neq 0$ and we may suspect that our model is not embedded in the 2+1-dimensional supergravity models claimed in Ref. [22]. In section 4, we will show that all possible regular self-dual solitons with rotational symmetry have boundary conditions $\phi_\infty = 0$ or 1 so that the model of our consideration shares properties of the vanishing cosmological constant, i.e., $V(|\phi| = 0; \phi_\infty = 0) = V(|\phi| = v; \phi_\infty = 1) = 0$. One more comment should be placed: For both cases to support regular self-dual solitons (nonperturbative spectra), the potential change due to G -dependent terms does not affect the mass spectra of perturbative gauge and scalar bosons. Therefore, equality of the gauge boson mass and the Higgs mass in broken phase ($\phi_\infty = 1$), still holds in curved spacetime, i.e., $m_{\text{gauge}} = m_{\text{Higgs}} = e^2 v^2 / |\kappa|$. The above properties imply that one of our present systems classified by an undetermined parameter ϕ_∞ may be obtained as the bosonic sector of some supergravity models.

3 Bogomolnyi Bound in Dual Formulation

We will construct duality transformed theory of our interest and see how the solitons arise in this framework. Let us work in the path integral formalism

$$Z = \int [dg_{\mu\nu}] [dA_\mu] [|\phi| d|\phi|] [d\Omega] e^{iS}, \quad (3.1)$$

where the action S was defined in Eq. (2.1). The interaction between the scalar and gauge fields is linearized by introducing an auxiliary field C_μ as follows

$$\begin{aligned} & \exp \left\{ i \int d^3x \sqrt{g} \frac{1}{2} g^{\mu\nu} |\phi|^2 (\partial_\mu \Omega - e A_\mu) (\partial_\nu \Omega - e A_\nu) \right\} \\ &= \int [dC_\mu] \prod_x \frac{g^{\frac{1}{4}}}{|\phi|^3} \exp i \int d^3x \sqrt{g} \left\{ -\frac{g^{\mu\nu}}{2|\phi|^2} C_\mu C_\nu + g^{\mu\nu} C_\mu (\partial_\nu \Omega - e A_\nu) \right\}. \end{aligned} \quad (3.2)$$

An auxiliary vector field C_μ is classically identified with the U(1) current by constraint equation. Contribution of the scalar phase $\Omega(x)$ is divided into two: one from topological excitations described by a multi-valued function $\Theta(x)$ and the other from fluctuations around superselected topological sector expressed by a single-valued function $\eta(x)$. Then the path integral measure for the scalar phase becomes

$$[d\Omega] = [d\Theta][d\eta], \quad (3.3)$$

and η -integration explains conservation of the U(1) current in the path integral formalism

$$\int [d\eta] \exp \left\{ i \int d^3x \sqrt{g} g^{\mu\nu} C_\mu \partial_\nu \eta \right\} \approx \frac{1}{\sqrt{g}} \delta(\nabla_\mu C^\mu). \quad (3.4)$$

Dual vector field H_μ is introduced such as

$$\int [dC_\mu] \frac{1}{\sqrt{g}} \delta(\nabla_\mu C^\mu) \cdots = \int [dH_\mu] [dC_\mu] \delta(\sqrt{g} C^\mu - \frac{\kappa}{e} \epsilon^{\mu\nu\rho\sigma} \partial_\nu H_\rho) \cdots, \quad (3.5)$$

where the scale κ/e is introduced for later convenience. The action at this stage is quadratic in both C_μ and A_μ so that the path integrals are Gaussian type. Integrating both A_μ and C_μ in a closed form, we obtain duality-transformed theory;

$$\begin{aligned} Z &= \int [g^{\frac{3}{4}} dg_{\mu\nu}] [dH_\mu] [|\phi|^{-2} d|\phi|] [d\Theta] \\ &\exp i \int d^3x \sqrt{g} \left\{ -\frac{1}{16\pi G} R - \frac{\kappa^2}{4e^2 |\phi|^2} g^{\mu\rho} g^{\nu\sigma} H_{\mu\nu} H_{\rho\sigma} - \frac{\kappa}{2} \frac{\epsilon^{\mu\nu\rho}}{\sqrt{g}} H_\mu \partial_\nu H_\rho \right. \\ &\quad \left. + \frac{\kappa}{2e} \frac{\epsilon^{\mu\nu\rho}}{\sqrt{g}} H_{\mu\nu} \partial_\rho \Theta + \frac{1}{2} g^{\mu\nu} \partial_\mu |\phi| \partial_\nu |\phi| - V(|\phi|) \right\}, \end{aligned} \quad (3.6)$$

where $H_{\mu\nu} = \partial_\mu H_\nu - \partial_\nu H_\mu$.

Equations of motion read

$$\frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu |\phi|) = -\frac{dV}{d|\phi|} + \frac{\kappa^2}{2e^2 |\phi|^3} g^{\mu\rho} g^{\nu\sigma} H_{\mu\nu} H_{\rho\sigma}, \quad (3.7)$$

$$\frac{1}{\sqrt{g}}\partial_\nu\left(\sqrt{g}\frac{H^{\mu\nu}}{|\phi|^2}\right) + \frac{e^2}{2\kappa}\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{g}}H_{\nu\rho} = \frac{e}{\kappa}\frac{\epsilon^{\mu\nu\rho}}{\sqrt{g}}\partial_\nu\partial_\rho\Theta, \quad (3.8)$$

$$G_{\mu\nu} = 8\pi GT_{\mu\nu}^D, \quad (3.9)$$

where the energy-momentum tensor is

$$\begin{aligned} T_{\mu\nu}^D = & \partial_\mu|\phi|\partial_\nu|\phi| - \frac{\kappa^2}{e^2|\phi|^2}g^{\rho\sigma}H_{\mu\rho}H_{\nu\sigma} \\ & - g_{\mu\nu}\left\{\frac{1}{2}g^{\alpha\beta}\partial_\alpha|\phi|\partial_\beta|\phi| - \frac{\kappa^2}{4e^2|\phi|^2}g^{\alpha\beta}g^{\gamma\delta}H_{\alpha\gamma}H_{\beta\delta} - V\right\}. \end{aligned} \quad (3.10)$$

In the Higgs phase with $|\phi| = v$, the interaction term between the Higgs field and the dual gauge field turns out to be Maxwell term so that the Chern-Simons Higgs theory is transformed to topologically massive gauge theory and thereby both theories support an odd-parity helicity-one photon of mass $e^2v^2/|\kappa|$ [16]. Even though the action in the duality-transformed theory (3.6) looks inappropriate to describe the symmetric phase of Chern-Simons scalar electrodynamics because of the singular factor $1/|\phi|^2$ in both the action and the path integral measure, this term plays an important role to produce nonperturbative spectra, e.g., nontopological solitons. For this, let us look at the Gauss' law in flat spacetime

$$\partial_i\left(\frac{E^i}{|\phi|^2}\right) + \frac{e^2}{\kappa}B = \pm\frac{2\pi e}{\kappa}\sum_{p=1}^n\delta^{(2)}(x^i - x_p^i). \quad (3.11)$$

Since the electric field term in the left-hand side of Eq. (3.11) has no long-range contribution in the Higgs phase, the magnetic flux must be quantized. If there exists a soliton of which the scalar field behaves like $|\phi| \sim |x^i|^{-\varepsilon}$ ($\varepsilon > 0$) for large $|x^i|$ in addition to the discrete magnetic flux, the magnetic flux has additional contribution to compete with that of the electric field term (3.11), which is not necessarily discrete. It is indeed the case of the Chern-Simons Higgs theory [8]. Since the gauge coupling e is inversely coupled to the Maxwell-like term, which describes interaction between the scalar and gauge fields, the strong coupling expansion is achievable when the nonpolynomial Higgs interaction is neglected [13]. Note that, though the classical gravity is not affected by the duality transformation, path integral measure for the gravitational field contains a nontrivial Jacobian factor which comes from gauge dynamics [17].

From now on we explore the static solitons given by solutions of the classical equations of motion. The duality transformation makes the system complicate by changing the first-order

Chern-Simons equation (2.4) to the second-order equation (3.8), however Eq. (3.8) is reduced to the first-order Chern-Simons equation if we replace the gauge field A_μ by the dual gauge field H_μ . It is due to the fact that a relation between the auxiliary field C_μ and the dual gauge field H_μ in the right-hand side of Eq. (3.5) is the form of classical Chern-Simons equation. Instead of solving the Euler-Lagrange equations (3.7)-(3.9), we derive a Bogomolnyi-type bound of the duality-transformed Einstein Chern-Simons Higgs theory again. Similar to the previous section, we define the energy as Euler invariant and assume the 00-component of stationary metric N^2 to be 1. Reshuffle of the terms of the energy gives us an expression of the energy such as

$$\begin{aligned}
E_D = & \int d^2x \sqrt{\gamma} \left\{ \frac{\kappa^2}{2e^2|\phi|^2} \gamma^{ij} \partial_i \left(H_0 \pm \frac{e}{2\kappa} (|\phi|^2 - v^2) \right) \partial_j \left(H_0 \pm \frac{e}{2\kappa} (|\phi|^2 - v^2) \right) \right. \\
& + \frac{1}{2} \left(\frac{\kappa}{e|\phi|} \tilde{H} \mp U \right)^2 \pm \frac{\kappa}{2e} |\phi|^2 \left[\frac{1}{\sqrt{\gamma}} \partial_i \left(\sqrt{\gamma} \gamma^{ij} \frac{1}{|\phi|^2} (\partial_j H_0 + \gamma^{kl} K_l \tilde{H}_{jk}) \right) \right. \\
& + \frac{e^2}{2\kappa} \frac{\epsilon^{ij}}{\sqrt{\gamma}} H_{ij} - \frac{e}{\kappa} \frac{\epsilon^{ij}}{\sqrt{\gamma}} \partial_i \partial_j \Theta \left. \right] - \left[\frac{\epsilon^{ij}}{\sqrt{\gamma}} \partial_i \left(K_j \left(\frac{\kappa}{e} H_0 + \frac{\tilde{H}}{|\phi|^2} \right) \right) \right] \\
& \pm \frac{\kappa}{e|\phi|} \tilde{H} \left(U - \frac{e^2}{2\kappa} |\phi| (|\phi|^2 - v^2) \right) \\
& + \left(W + \frac{e}{2} K \left(\pm (|\phi|^2 - v^2) H_0 + \frac{ev^4}{2\kappa} C_D \right) - \frac{3}{32\pi G} K^2 \right) \left. \right\} \\
& \mp \frac{ev^2}{4} \int d^2x \epsilon^{ij} H_{ij} \\
& + \int d^2x \partial_i \left[\pm \frac{ev^2}{2} \epsilon^{ij} K_j \left(H_0 \mp \frac{ev^2}{2\kappa} C_D \right) \partial_i \mp \frac{\kappa}{2e} \sqrt{\gamma} \gamma^{ij} \partial_j H_0 \right],
\end{aligned} \tag{3.12}$$

where $\tilde{H} = \frac{\epsilon^{ij}}{2\sqrt{\gamma}} \tilde{H}_{ij} = \frac{\epsilon^{ij}}{2\sqrt{\gamma}} (H_{ij} + K_i \partial_j H_0 - K_j \partial_i H_0)$, and C_D is an integration constant.

Similar to the procedure done in the section 2, we obtain the Bogomolnyi-type bound for the duality-transformed Einstein Chern-Simons Higgs theory. Square brackets in the second and third lines in Eq. (3.12) vanish by use of the equations for the gauge fields (3.8). Let us assume that configurations satisfy

$$H_0 = \mp \frac{e}{2\kappa} (|\phi|^2 - v^2). \tag{3.13}$$

Then we can fix C_D as $1 - \phi_\infty^2$ from the condition that the objects of our interest have spin J and the boundary terms in the last line of Eq. (3.12) vanish. With the aid of the solutions of 0i-components of the Einstein equations, we choose a specific form of the scalar potential

to let the terms in the fourth and fifth lines be zero. Hence the Bogomolnyi-type bound is attained for the solutions of the following equation

$$\tilde{H} = \pm \frac{e|\phi|}{\kappa} U = \pm \frac{e^3}{2\kappa} |\phi|^2 (|\phi|^2 - v^2). \quad (3.14)$$

Replacing the gauge and gravitational fields in Eq. (3.14) by use of the relations obtained above, we have the same Bogomolnyi equations in Eq. (2.23) and Eq. (2.24). Among the remaining equations, solutions of the Bogomolnyi equation satisfy ij -components of the Einstein equations (3.9) and reproducibility of the scalar equation (3.7) determines the integration constant introduced by $0i$ -component of the Einstein equations.

4 Soliton Solutions

In this section we explore soliton solutions by examining the equations of motion, i.e., one is Bogomolnyi equation (2.23) and the other is the remaining Einstein equation (2.24). We restrict our interest to rotationally symmetric configurations satisfying the following three conditions;

1. nonsingular solutions of equations of motion,
2. solutions of finite energy which is proportional to the magnetic flux, and
3. the underlying manifold defined by the solution does not include curvature singularity.

The stationary metric (2.2) compatible with rotationally symmetric solutions becomes

$$ds^2 = \left(dt + \frac{\kappa}{e^2 v^2} \frac{d \ln \psi(r)}{dr} r d\theta \right)^2 - \left(\frac{\kappa}{e^2 v^2} \right)^2 b(r) (dr^2 + r^2 d\theta^2), \quad (4.1)$$

where $r = \sqrt{\tilde{x}^i \tilde{x}^i}$. With the help of a gauge transformation, ansatz for the scalar field is

$$\phi = v f(r) e^{-in\theta}, \quad (4.2)$$

and thereby a component of the metric b (2.21) becomes

$$b(r) = F \left[\frac{f^2 e^{-(f^2-1)} \psi^{1-\phi_\infty^2}}{r^{2n}} \right]^{\tilde{G}}, \quad (4.3)$$

where F is an undetermined constant due to the harmonic function $h(\tilde{z})$. Under this (r, θ) -coordinate, the equations of motion (2.23) - (2.24) become

$$\frac{1}{r} \frac{d}{dr} r \frac{d}{dr} \ln \frac{f^2}{r^{2n}} = -\frac{F\tilde{G}}{2} \left[\frac{f^2 e^{-(f^2-1)} \psi^{1-\phi_\infty^2}}{r^{2n}} \right]^{\tilde{G}} (f^2 - 1) \left[f^4 - 2\left(1 + \frac{1}{\tilde{G}}\right) f^2 + \phi_\infty^2 \right], \quad (4.4)$$

$$\frac{1}{r} \frac{d}{dr} r \frac{d}{dr} \ln \psi = -\frac{F\tilde{G}}{2} \left[\frac{f^2 e^{-(f^2-1)} \psi^{1-\phi_\infty^2}}{r^{2n}} \right]^{\tilde{G}} (f^4 - 2f^2 + \phi_\infty^2). \quad (4.5)$$

To obtain nonsingular solution at the origin, Eq. (4.2) fixes the boundary condition of the scalar field at $r = 0$

$$nf(r=0) = 0 \quad (4.6)$$

which implies the leading behavior of scalar field for small r

$$f \sim f_0 r^n, \quad (4.7)$$

and Eq. (4.5) gives the boundary condition of ψ at $r = 0$

$$\psi(r=0) = \psi_0, \quad (4.8)$$

where ψ_0 is an arbitrary positive constant.

Rewriting the magnetic flux (2.11) in terms of f and ψ

$$\Phi = \pm \frac{2\pi}{e} \left[n - r \frac{d \ln f}{dr} - \frac{1}{2} (1 - f^2) r \frac{d \ln \psi}{dr} \right] \Big|_{r \rightarrow \infty} \quad (4.9)$$

and the angular momentum (2.12) as

$$J = -\frac{\pi\kappa}{2e^2} \int_0^\infty dr r F \left(\frac{f^2 e^{-(f^2-1)} \psi^{1-\phi_\infty^2}}{r^{2n}} \right)^{\tilde{G}} (f^4 - 2f^2 + \phi_\infty^2) \quad (4.10)$$

$$= -\frac{\pi\kappa}{e^2 \tilde{G}} r \frac{d}{dr} \ln \psi \Big|_{r=\infty}, \quad (4.11)$$

we notice that, from Eq. (4.11), spinning objects have long range term of ψ for large r , specifically $\psi \sim r^{-2\alpha}$, where α is an arbitrary number and it also contributes to the magnetic flux (4.9) if $\phi_\infty \neq 1$. The energy (2.9) can also be expressed in terms of f and ψ such as

$$\begin{aligned} E &= \pi v^2 \int_0^\infty dr r \left\{ \left(\frac{df}{dr} \right)^2 + \frac{F}{4} \left(\frac{f^2 e^{-(f^2-1)} \psi^{\phi_\infty^2-1}}{r^{2n}} \right)^{\tilde{G}} \left[f^2 (f^2 - 1)^2 - \frac{\tilde{G}}{2} (f^4 - 2f^2 + \phi_\infty^2)^2 \right] \right\} \\ &= \frac{v^2}{2} |e\Phi|. \end{aligned} \quad (4.12)$$

Eqs. (4.12) - (4.10) give possible boundary conditions of the scalar amplitude at $r = \infty$ for a given α

$$\frac{|\phi|}{v} = \begin{cases} 0 \text{ or } 1, & \text{when } \tilde{G}[n + \alpha(1 - \phi_\infty^2)] \leq 1 \\ \text{arbitrary,} & \text{when } \tilde{G}[n + \alpha(1 - \phi_\infty^2)] > 1. \end{cases} \quad (4.13)$$

About the geometry of two dimensional space Σ described by the γ_{ij} part of Eq. (2.2) (or by the $b(r)$ of Eq. (4.1)), various manifolds are characterized by the area of space defined by $\mathcal{A} = 2\pi \int_0^\infty dr r b(r)$. Suppose that the asymptotic behavior of the scalar amplitude as $|\phi|/v \sim r^{-\varepsilon}$ where ε is related to α , the above categories are distinguished by the value of $\tilde{G}[n + \varepsilon + (1 - \phi_\infty^2)\alpha]$, i.e., whether it is less than or equal or greater than 1. Additional quantities characterizing the property of Σ are the radial distance from the origin $\rho(r) = (e^2 v^2 / |\kappa|) \int_0^r dr' \sqrt{b(r')}$ and the circumference $l(r) = (2\pi e^2 v^2 / |\kappa|) r \sqrt{b(r)}$. When $\varepsilon = 0$, the above expression of the scalar amplitude $|\phi|/v \sim r^{-\varepsilon}$ corresponds to a nonzero finite ϕ_∞ . Rapid decay of the scalar amplitude is depicted as $\varepsilon \rightarrow \infty$. So this assumption may not lose any generality. In this section we will obtain the rotationally symmetric solutions carrying magnetic flux (or charge) and spin. Underlying spatial manifold Σ will comprise three types; cone, cylinder, and two sphere. We address this question separately for these three cases.

From now on, let us examine precisely all possible rotationally symmetric solutions of the Bogomolnyi equations (4.4)-(4.5). When $\phi_\infty = 1$, Eq. (4.4) for f is decoupled from the equation for ψ (4.5). Thus it may be convenient to investigate the soliton solutions for the cases separately by whether ϕ_∞ is equal to one or not.

(a) $\phi_\infty = 1$

We consider the case $\phi_\infty = 1$ by dividing it into two categories: $\tilde{G}n \neq 1$ and $\tilde{G}n = 1$. When $\tilde{G}n$ is smaller than one, we introduce variables u and R such that

$$u = \ln f^2 \quad (4.14)$$

and

$$R = \frac{r^{1-\tilde{G}n}}{1-\tilde{G}n}. \quad (4.15)$$

Then the Bogomolnyi equation (4.4) is rewritten as

$$\frac{d^2 u}{dR^2} = -\frac{dU_{\text{eff}}}{du} - \frac{1}{R} \frac{du}{dR}, \quad (4.16)$$

where U_{eff} is given by

$$U_{\text{eff}}(u; \phi_\infty = 1) = -\frac{F}{\tilde{G}} \exp[-\tilde{G}(e^u - u - 1)](e^u - 1)^2. \quad (4.17)$$

As shown in Fig. 2-(a), $U_{\text{eff}}(u; \phi_\infty = 1)$ has two local minima at $u = \ln[(\tilde{G} + 1 \pm \sqrt{2\tilde{G} + 1})/\tilde{G}]$ and three maxima at $u = 0, \pm\infty$. If we interpret u as the position of a hypothetical particle with unit mass and R as time, Eq. (4.16) is Newton's equation for 1-dimensional motion of the hypothetical particle moving in the potential U_{eff} and subject to a time-dependent friction, $-(du/dR)/R$. For $n \neq 0$ cases, the particle also receives an impact at $R = 0$ from the delta function term, i.e., $[2n/(1 - \tilde{G}n)^{1/(1-\tilde{G}n)}]\delta(R^{1/(1-\tilde{G}n)})$ from Eq. (2.23). The energy of the hypothetical particle $\mathcal{E}(R)$ is defined by

$$\mathcal{E}(R) = \frac{1}{2} \left(\frac{du}{dR} \right)^2 + U_{\text{eff}}(u), \quad (4.18)$$

and it is a monotonically decreasing function of R because of the friction. When $n = 0$, trivial solution $u = 0$ is the unique solution which describes two dimensional flat space. We now show that, only when $0 < \tilde{G}n < 1$, there always exists a finite energy solution whose base manifold Σ is an asymptotic cone.

For this we prove that, for a suitably-chosen initial parameter f_0 , we can obtain a motion of the hypothetical particle such that it starts at negative infinity with the initial velocity given by Eq. (4.7) and stops at $u = 0$ at $R = \infty$.

For $r \rightarrow 0$, the behavior of the scalar field turns out to be

$$f \approx \begin{cases} f_0 r^n \left[1 + \frac{F}{16} \tilde{G} e^{\tilde{G}} f_0^{2\tilde{G}} r^2 - \frac{F}{1024} (1 + \tilde{G}) e^{2\tilde{G}} \left(16(2 + \tilde{G}) e^{-\tilde{G}} f_0^{2(1-\tilde{G})} - \tilde{G}^2 F \right) f_0^{4\tilde{G}} r^4 + \dots \right], & \text{when } n = 1, \\ f_0 r^n \left[1 + \frac{F}{16} \tilde{G} e^{\tilde{G}} f_0^{2\tilde{G}} r^2 - \frac{F^2}{512} (1 + \tilde{G}) e^{2\tilde{G}} f_0^{4\tilde{G}} r^4 + \dots \right], & \text{when } n \geq 2. \end{cases} \quad (4.19)$$

Let us fix an arbitrarily-large number R_0 for R in Eq. (4.15). If we choose f_0 sufficiently small so that $f_0^{\tilde{G}} r^2 \ll 1$ then the terms higher than second-order in Eq. (4.19) can be neglected for $R < R_0$. If we substitute this approximated solution (4.19) into Eq. (4.18) then the energy of the particle $\mathcal{E}(R_0)$ in Eq. (4.18) is estimated by

$$\mathcal{E}(R_0) \approx \frac{n^2}{2(1 - \tilde{G}n)^2 R_0^2} + \frac{F}{2} \tilde{G} n (1 - \tilde{G}n)^{\frac{2\tilde{G}n}{1-\tilde{G}n}} e^{\tilde{G}} f_0^{2\tilde{G}} \left[\frac{1}{R_0} - \frac{2}{\tilde{G}^2 n} \right] R_0^{\frac{2\tilde{G}n}{1-\tilde{G}n}}. \quad (4.20)$$

Let us choose R_0 and f_0 such that the square bracket in the right-hand side of Eq. (4.20) negative and the absolute value of this second term is larger than the first term. Furthermore, by taking f_0 small enough, additional positive contribution from the negative higher order

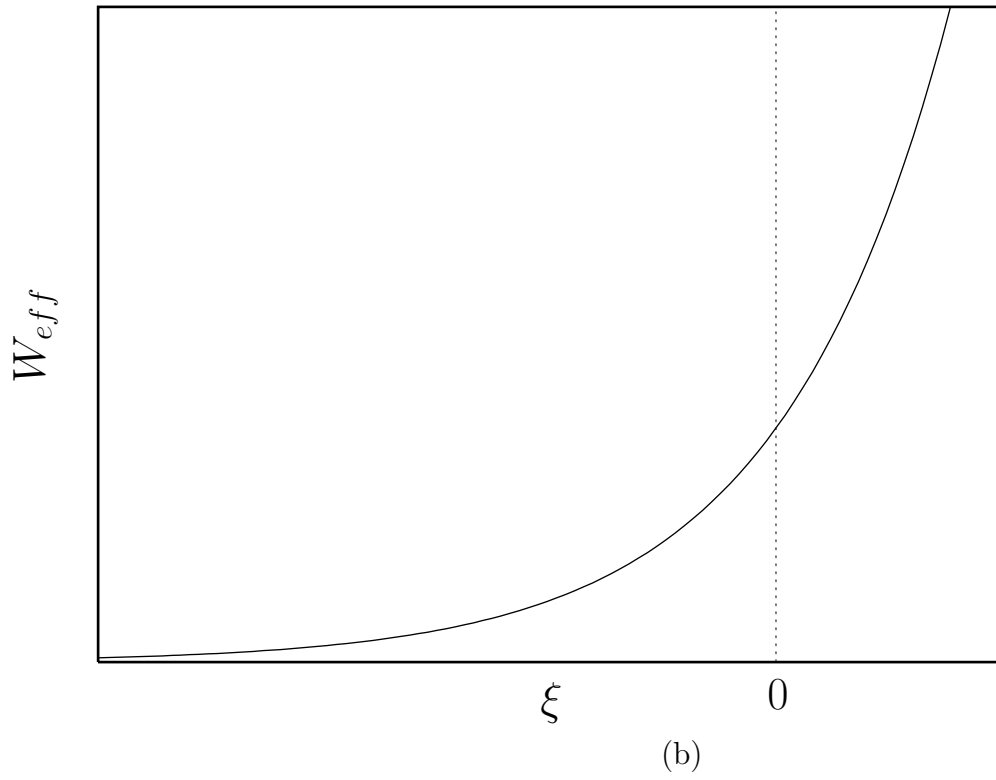
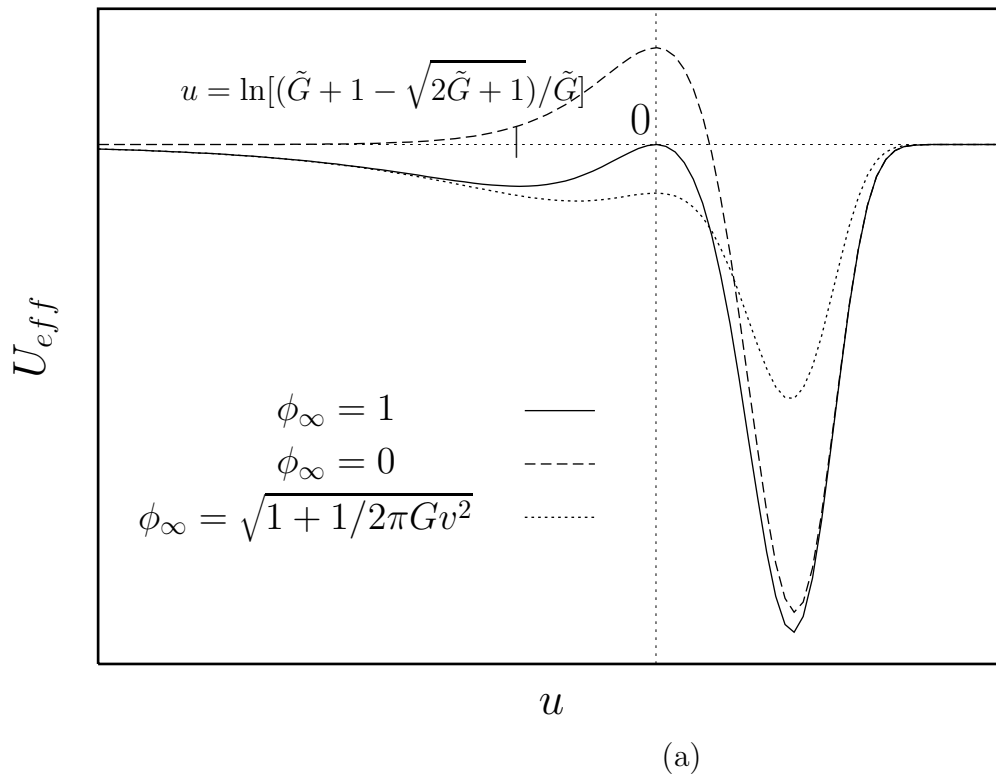


Figure 2: (a) Typical shapes of U_{eff} : The solid, dashed, and dotted lines correspond to $\phi_\infty = 1, 0$, and $\sqrt{1 + 1/2\pi G v^2}$. (b) A typical shape of W_{eff} .

terms does not change the sign of $\mathcal{E}(R_0)$. Then $\mathcal{E}(R_0)$ is negative, which means $\mathcal{E}(R_0) < U_{\text{eff}}(0)$. Since the particle energy \mathcal{E} decreases further as $R > R_0$ becomes large due to friction, the hypothetical particles has a turning point for sufficiently small f_0 and stops finally at a local minimum of the effective potential $u(R = \infty) = \ln[(\tilde{G} + 1 - \sqrt{2\tilde{G} + 1})/\tilde{G}]$ after some oscillation around it. For the large f_0 and the sufficiently small R_0 which makes Taylor expansion in Eq. (4.19) valid, it is easy to notice that, for $R > R_0$,

$$u(R) > 2 \ln f_0 r^n + \frac{U'_M}{4}(R_0^2 - R^2) + \frac{U'_M}{2}R_0^2 \ln R_0/R, \quad (4.21)$$

where $U'_M = \max_{-\infty < u \leq 0} (dU_{\text{eff}}/du) > 0$. The right-hand side of Eq. (4.21) has maximum value at $R = R_1 \equiv \left[R_0^2 + \frac{4n}{U'_M[1-\tilde{G}(n+\tilde{M})]} \right]^{1/2}$ and then $u(R_1)$ satisfies, for a small R_0 ,

$$u(R_1) > \frac{n}{1 - \tilde{G}n} \left[\ln \frac{4n(1 - \tilde{G}n)}{U'_M} - 1 \right] + 2 \ln f_0. \quad (4.22)$$

Therefore if we choose f_0 sufficiently large as assumed previously, then $u(R_1) > 0$, i.e., the particle went over the hilltop of the potential at $u = 0$. From these results, continuity now guarantees existence of a vortex solution connecting the boundary values, $u(0) = -\infty$ and $u(\infty) = 0$, for an appropriate f_0 . This completes the proof and a specific example $f(= |\phi|/v)$ is shown in Fig. 3.

Next we discuss that this solution constitutes an asymptotic cone for $\tilde{G}n < 1$ case, however there exists no solution for $\tilde{G}n > 1$ case. When $\tilde{G}n < 1$, the radial distance ρ and the circumference l diverge as r goes to infinity. The solutions of Eq. (4.4) approach their boundary values exponentially

$$f \approx 1 - f_\infty K_0((1 - \tilde{G}n)R), \quad (4.23)$$

where f_∞ is a constant determined by the proper behavior of the fields near the origin. Thus the asymptotic structure of Σ is a cone with deficit angle $\delta = 2\pi\tilde{G}n$ which is flat. With the help of Eqs. (4.14) -(4.15), introduction of a new variable such as

$$\chi = \ln \psi \quad (4.24)$$

rewrites the equation (4.5) as another Newton's equation for a particle whose position is χ

$$\frac{d^2\chi}{dR^2} = -\frac{F}{2}\tilde{G}e^{-\tilde{G}(e^u - u - 1)}(e^u - 1)^2 - \frac{1}{R}\frac{d\chi}{dR}. \quad (4.25)$$

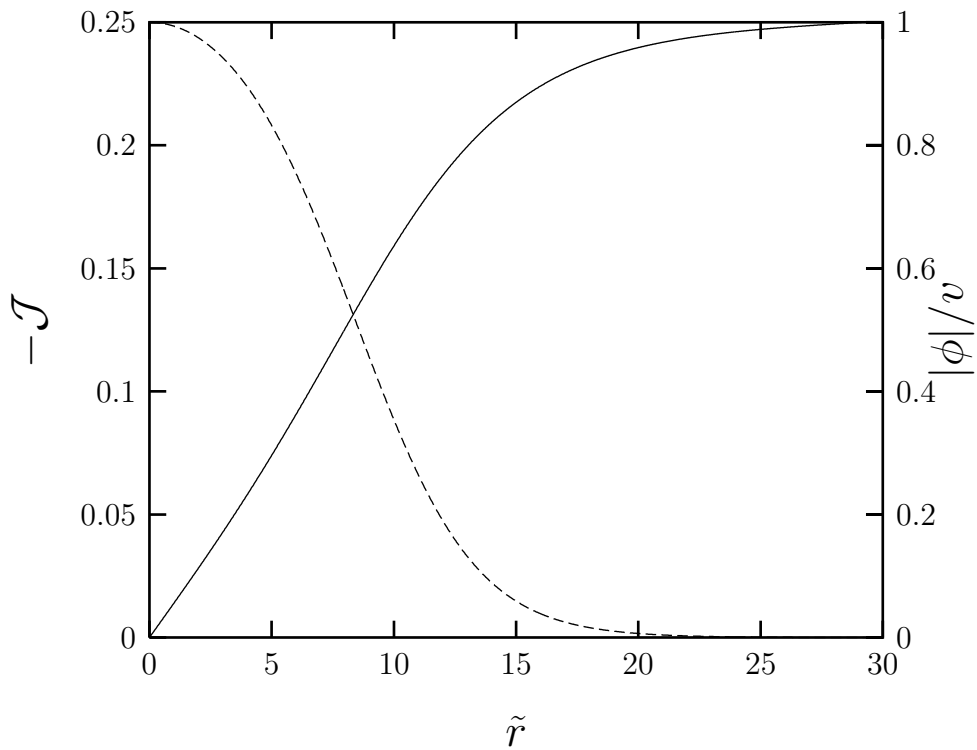


Figure 3: Plot of rotationally symmetric solution for $\phi_\infty = 1$. Parameters chosen in the figures are: $n = 1$, $F = 1$, and $\tilde{G} = 1/2$ for curved spacetime. The solid and dashed lines correspond to $|\phi|/v$ and \mathcal{J} .

The external force in the right-hand side of Eq. (4.25) is always negative so that a particle starts at $\chi = \ln \psi_0$ with zero initial velocity and goes to negative infinity as it decelerates to zero for large R . Since the external force term in the right-hand side of Eq. (4.25) decays exponentially for large R , the leading term of asymptotic solution for χ is obtained by solving the linearized equation (4.25). Suppose that $\psi \sim r^{-2\alpha}$ ($\alpha \geq 0$) for large r , these topological vortices carry quantized magnetic flux (4.9)

$$\Phi = \pm \frac{2\pi n}{e} \quad (4.26)$$

and angular momentum (4.11)

$$J = \frac{2\pi\kappa}{e^2\tilde{G}}\alpha. \quad (4.27)$$

Note that the spin is arbitrary in curved space which is different from the result of flat spacetime, i.e., $J \propto n^2$ in flat spacetime.

Let us consider the case that $\tilde{G}n$ is larger than 1. Suppose that there exists a solution when $\tilde{G}n > 1$. Then the radial distance $\rho(r = \infty)$ is finite and the circumference $l(r = \infty)$ vanishes since the metric $b(r)$ decreases faster than $1/r^2$ for large r , which means that the spatial manifold Σ is bounded. Since the Euler number given in Eq. (2.26) must be nonnegative due to the fast decaying property of the solution, two dimensional sphere S^2 is unique candidate. From now on we call the point which corresponds to $r = 0$ “the south pole” on S^2 and that which corresponds to $r = \infty$ “the north pole” on S^2 . The configuration of the scalar field ϕ at the north pole does not vanish as $\phi(r = \infty) = ve^{in\theta}$. Therefore the scalar field ϕ is not well-defined at the north pole and there is no regular solution in this case.

Let us study the case when $\phi_\infty = 1$ and $\tilde{G}n = 1$. We introduce R as

$$R = \ln r, \quad (-\infty < R < \infty) \quad (4.28)$$

which reflects the scale symmetry ($r \rightarrow \lambda r$) of the Bogomolnyi equation (4.4) at this critical value. Then Eq.(4.4) takes the same form as Eq. (4.16) without friction term. Similar to the previous cases, a hypothetical particle moves under a conservative force only and hence the Bogomolnyi equation can be integrated to a first-order equation (4.18) whose energy \mathcal{E} of the hypothetical particle is conserved in this case. The initial particle energy $\mathcal{E}(R = -\infty)$

is determined by the initial behavior of f (4.7)

$$\mathcal{E} = \frac{1}{2} \left(\frac{du}{dR} \right)^2 \Big|_{R=-\infty} = 2n^2. \quad (4.29)$$

Here if we recall that three maxima at $u = 0, \pm\infty$ are degenerate, we can easily notice that there is no such particle motion which satisfies $u = 0$ as $R \rightarrow \infty$. Therefore it completes the nonexistence of soliton solution when $\phi_\infty = 1$ and $\tilde{G}n = 1$.

When $\phi_\infty \neq 1$, f and ψ are described by the coupled equations (4.4) and (4.5). As has done in the case of $\phi_\infty = 1$, we introduce a new variable

$$\xi = \sqrt{|1 - \phi_\infty^2|} \ln \psi \quad (4.30)$$

with u and R in Eqs. (4.14), (4.15) and (4.28). Then the Bogomolnyi equations (4.4)-(4.5) become

$$\frac{d^2 u}{dR^2} = -\frac{\partial V_{\text{eff}}}{\partial u} - \frac{1}{R} \frac{du}{dR}, \quad (4.31)$$

$$\frac{d^2 \xi}{dR^2} = -\frac{\partial(\mp V_{\text{eff}})}{\partial \xi} - \frac{1}{R} \frac{d\xi}{dR}, \quad (4.32)$$

where \mp in Eq. (4.32) denotes $-$ for $\phi_\infty < 1$ cases and $+$ for $\phi_\infty > 1$ cases. V_{eff} is

$$\begin{aligned} V_{\text{eff}}(u, \xi; \phi_\infty) &= U_{\text{eff}}(u; \phi_\infty) W_{\text{eff}}(\xi; \phi_\infty) \\ &= \left[-\frac{F}{\tilde{G}} e^{-\tilde{G}(e^u - u - 1)} (e^{2u} - 2e^u + \phi_\infty^2) \right] \times \left[e^{\tilde{G}\sqrt{|1 - \phi_\infty^2|}\xi} \right]. \end{aligned} \quad (4.33)$$

Note that Eq. (4.31) contains an impact term at a boundary value of R which corresponds to $r = 0$ if $n \neq 0$, and the friction term in both equations (4.31)-(4.32) vanishes if $\tilde{G}n = 1$, which reflects the scale symmetry ($r \rightarrow \lambda r$) of the Bogomolnyi equations (4.4)-(4.5).

We continue to use the terminology of Newton's equation for the two-dimensional motion (u, ξ) of a particle when $\phi_\infty > 1$ and for one-dimensional motion of two interacting particles u and ξ , when $\phi_\infty < 1$. The boundary values of u at $R = \infty$ can be read from shapes of the effective potential U_{eff} . In order to constitute a soliton configuration, minimum requirement to the boundary value of the scalar field, $v\phi_\infty$, at spatial infinity is to be an extremum of the effective potential U_{eff} . Specifically, from $dU_{\text{eff}}/du = 0$ and $e^{u(R=\infty)} = \phi_\infty^2$, we can easily read that possible values of ϕ_∞ are $0, 1, \sqrt{1 + 2/\tilde{G}}$ and ∞ . Since $\phi_\infty = 1$ case was

already analyzed, let us discuss the other cases. One point on the cosmological constant should be noted: We recall that $\phi_\infty = 0, 1, 1/\sqrt{\pi G v^2}$ make the potential vanish, that is, $V(|\phi| = v\phi_\infty) = 0$. If they exist, regular solutions satisfying the other boundary conditions, $\phi_\infty = \sqrt{1 + 2/\tilde{G}}$ or $\phi_\infty = \infty$, imply self dual solitons in curved spacetime with a nonvanishing cosmological constant.

(b) $\phi_\infty = 0$

For this case the scalar field approaches the symmetric local minimum for large r as shown in Fig. 1 (dashed line). Then the Chern-Simons gauge field is topological field, i.e., it is no propagating degree, and charged meson is massive, $m_{\text{meson}} = (e^2 v^2 / 2|\kappa|)$. All the solitonic excitations are the nontopological solitons which are marginally stable since $E = m_{\text{meson}} Q = (e^2 v^2 / 2|\kappa|) Q$. We consider those solutions with nonzero vorticity ($n \neq 0$) separately from those solutions with no vorticity ($n = 0$).

(b-i) $n = 0$

As mentioned previously, we regard Eqs. (4.31) -(4.32) as Newton's equations for two particles of which the positions are $u(R)$ and $\xi(R)$, respectively. Here $n\tilde{G} = 0$ and then R is in fact equal to r as shown in Eq. (4.15). Since the force, $-\partial V_{\text{eff}}/\partial u$, applied to one particle, of which the position is u , is negative definite for u in the region $-\infty < u < 0$ and for all ξ , this particle starts out at time zero, $R = 0$, at a point u_0 ($-\infty < u_0 < 0$) and approaches to negative infinity $u = -\infty$, at infinite time $R = \infty$. Similarly, since the force $-\partial(-V_{\text{eff}})/\partial \xi$ applied to the other particle, of which the position is, ξ is positive definite for any ξ and u in the region $-\infty < u < 0$, this particle initially starts out at a point ξ_0 and finally goes to positive infinity, $\xi = \infty$.

Near the origin power series solutions give

$$f(r) \approx g_0 \left(1 - \frac{F}{16} g_0^{2(1+\tilde{G})} (1 - g_0^2) [2(1 + \tilde{G}) - \tilde{G} g_0^2] e^{\tilde{G}(1-g_0^2)} \alpha_0^{\tilde{G}} r^2 + \dots \right), \quad (4.34)$$

$$\psi(r) \approx \alpha_0 \left[1 + \frac{F}{8} \tilde{G} g_0^{2(1+\tilde{G})} (2 - g_0^2) e^{\tilde{G}(1-g_0^2)} \alpha_0^{\tilde{G}} r^2 + \dots \right]. \quad (4.35)$$

Eqs. (4.4)-(4.5) do not constrain both g_0 ($0 < g_0 < 1$) and α_0 , however if $g_0 > 1$, both $f(r)$ and $\psi(r)$ will be monotonically-increasing functions of r and hence the boundary condition

at $r = \infty$ cannot be met. It is consistent with the argument given in terms of Newton's equations for the region of u_0 ($-\infty < u_0 < 0$). Eqs. (4.4)-(4.5) imply that long distance behaviors of the scalar field f and the off-diagonal component of metric ψ should be $f(r) \sim r^{-\varepsilon}$ and $\psi(r) \sim r^{2\alpha}$ so that Eq. (4.3) gives $b(r) \sim r^{-2\tilde{G}(\varepsilon-\alpha)}$. Positivity of the energy (2.10) restricts the values of α and ε to satisfy $\varepsilon - \alpha > 0$. The precise forms of power series solutions for large r are

$$f(r) \approx g_\infty r^{-\varepsilon} \left[1 - \frac{F(1+\tilde{G})e^{\tilde{G}}}{8[\tilde{G}(\varepsilon-\alpha) + \varepsilon - 1]^2} g_\infty^{2(1+\tilde{G})} \alpha_\infty^{\tilde{G}} r^{-2(\tilde{G}(\varepsilon-\alpha)+\varepsilon-1)} + \dots \right], \quad (4.36)$$

$$\psi(r) \approx \alpha_\infty r^{2\alpha} \left[1 - \frac{F\tilde{G}e^{\tilde{G}}}{[\tilde{G}(\varepsilon-\alpha) + \varepsilon - 1]^2} g_\infty^{2(1+\tilde{G})} \alpha_\infty^{\tilde{G}} r^{-2(\tilde{G}(\varepsilon-\alpha)+\varepsilon-1)} + \dots \right], \quad (4.37)$$

where g_∞ and α_∞ are the constants determined by the behavior of the fields near the origin. It is of our interest to determine the lower bound of $\tilde{G}(\varepsilon - \alpha) + \varepsilon - 1$ by examining the correspondence between the short-distance and long-distance behaviors of the solutions, i.e., (g_0, α_0) and (ε, α) . Since f becomes small near the origin for a sufficiently-small g_0 and an arbitrary α_0 , we can expand the right-hand side of Eqs. (4.4)-(4.5) for the leading order of f . If we define g as $g(r) = Fe^{\tilde{G}}(1 + 2\tilde{G})f^{2(1+\tilde{G})}\psi^{\tilde{G}}$, then the approximate equation for g is the Liouville equation. An exact solution satisfying the boundary condition at the origin is

$$g(r) = \frac{g_0^{2(1+\tilde{G})} \alpha_0^{\tilde{G}}}{(1 + \frac{1}{8}g_0^{2(1+\tilde{G})} \alpha_0^{\tilde{G}} r^2)^2}, \quad (4.38)$$

which is small enough for all r and approaches to the exact solution of Eqs. (4.4)-(4.5) as $g_0 \rightarrow 0$. Comparing the solution (4.38) with those in Eqs. (4.36)-(4.37), we obtain a lower bound for ε and α , i.e., $\tilde{G}(\varepsilon - \alpha) + \varepsilon - 1 \geq 1$, and it is consistent with regularity of the power series solutions (4.36)-(4.37). As $g_0 \rightarrow 0$, the magnetic flux becomes $\Phi = \pm(2 - \varepsilon)/2eGv^2$ and it implies a constraint for ε , i.e., $\varepsilon \leq 2$.

Furthermore, another category in examining the solutions is to know that the area \mathcal{A} of spatial manifold Σ is finite or infinite. When $\tilde{G}(\varepsilon - \alpha) < 1$, the area \mathcal{A} , the radial distance at infinity $\rho(r = \infty)$, and the circumference at infinity $l(r = \infty)$ are infinite. Then the asymptotic structure of Σ is also a cone with deficit angle $\delta = 8\pi^2 Gv^2(\varepsilon - \alpha)$. Though the global structure of the cone is the same as that of vortices of $\phi_\infty = 1$ case, these manifolds have long-range tails for large r as shown in Fig. 4-(a) and the shape of the angular

momentum density (see Fig. 4-(b)) shows that it is accumulated near the origin. Since the circumference at infinity $l(r = \infty)$ is finite when $\tilde{G}(\varepsilon - \alpha) = 1$, the asymptotic structure of Σ is a cylinder. When $\tilde{G}(\varepsilon - \alpha) > 1$, the space Σ is bounded manifold and then a two sphere is unique candidate. The Euler number fixes $\tilde{G}(\varepsilon - \alpha) = 2$ that is $b(r) \sim r^{-4}$. Then the regularity at the north pole ($r = \infty$) requires that the scalar field behaves as $f \sim s^n$ for $s \sim 0$, where s is radial coordinate in the new coordinate system whose origin is at the north pole. However, since the scalar field ϕ does not vanish at the south pole which corresponds to the point $r = 0$ or equivalently $s = \infty$, it is not well-defined. Specifically, $\phi = |\phi|e^{in\theta} = g_0 e^{in\theta}$ at the north pole. Therefore there is no regular solution when $\tilde{G}(\varepsilon - \alpha) > 1$.

Since the solutions with $|\phi|(\infty) = 0$ are nontopological solitons, they are characterized by the U(1) charge (or equivalently the magnetic flux) $Q(= -\kappa\Phi) = \mp(2\pi\kappa/e)\left[\varepsilon + (e^2\tilde{G}/2\pi\kappa)J\right]$ and the angular momentum $J = -(2\pi\kappa/e^2\tilde{G})\alpha$, which need not be quantized. Note that the sign of angular momentum for these nontopological solitons is opposite to that of vortices as given in Eq. (4.27) as in the case of flat spacetime. In case of the cylinder the magnetic flux has a fixed value $\Phi = 2\pi\kappa/e\tilde{G}$ for a given set of parameters.

(b-ii) $n \neq 0$

For $n \neq 0$ solutions, the motion of a hypothetical particle whose position is depicted by ξ coordinate resembles that of the $n = 0$ solution since it follows the same equation of motion (4.32) and the starting point is also an arbitrary constant ξ_0 . On the other hand, the motion of the other hypothetical particle whose position is u coordinate is different from $n = 0$ solution due to the vorticity. Specifically, the particle should start out at negative infinity, $u = -\infty$, with an initial velocity given by Eq. (4.7) which is induced by receiving an initial impact at time corresponding to $r = 0$, and then it turns at an appropriate position u_{\max} at a certain time, and finally returns to $u = -\infty$ at time $R(r)|_{r=0}$. We now argue that there exist such nontopological vortex solutions by showing the existence of turning point u_{\max} between $-\infty$ and 0.

For small r , power series solutions are

$$f(r) \approx h_0 r^n \left[1 - \frac{F}{8(n+1)^2} (1 + \tilde{G}) e^{\tilde{G}} h_0^{2(1+\tilde{G})} \beta_0^{\tilde{G}} r^{2(n+1)} + \dots \right], \quad (4.39)$$

$$\psi(r) \approx \beta_0 \left[1 + F \tilde{G} e^{\tilde{G}} h_0^{2(1+\tilde{G})} \beta_0^{\tilde{G}} r^{2(n+1)} + \dots \right]. \quad (4.40)$$

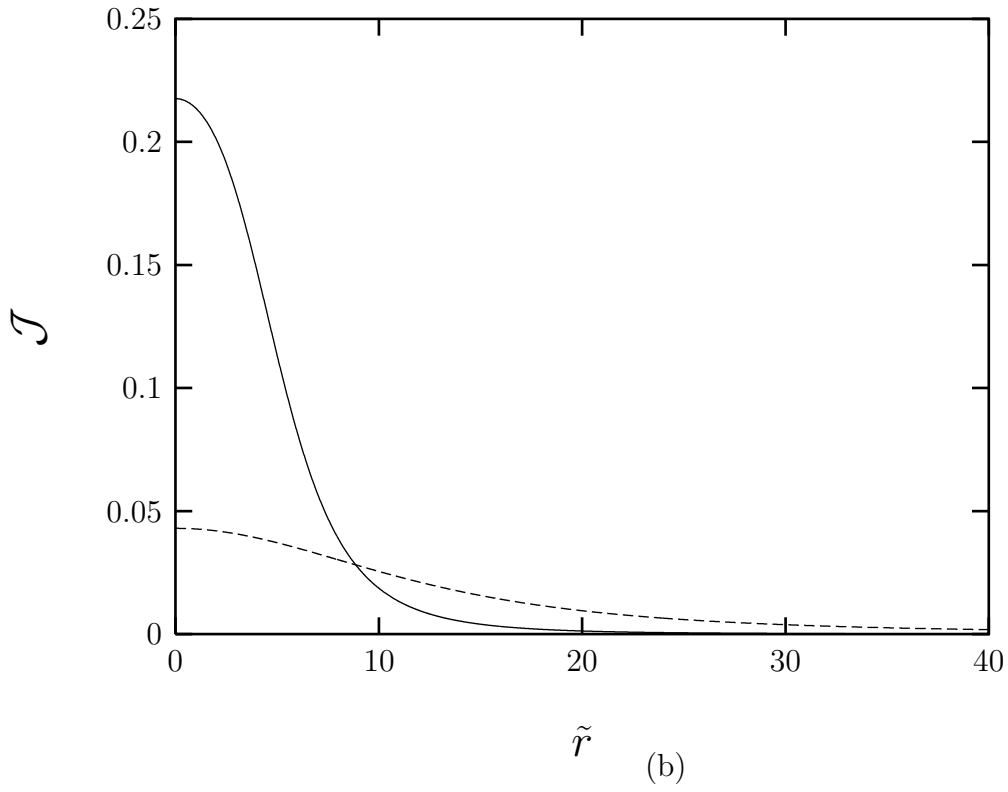
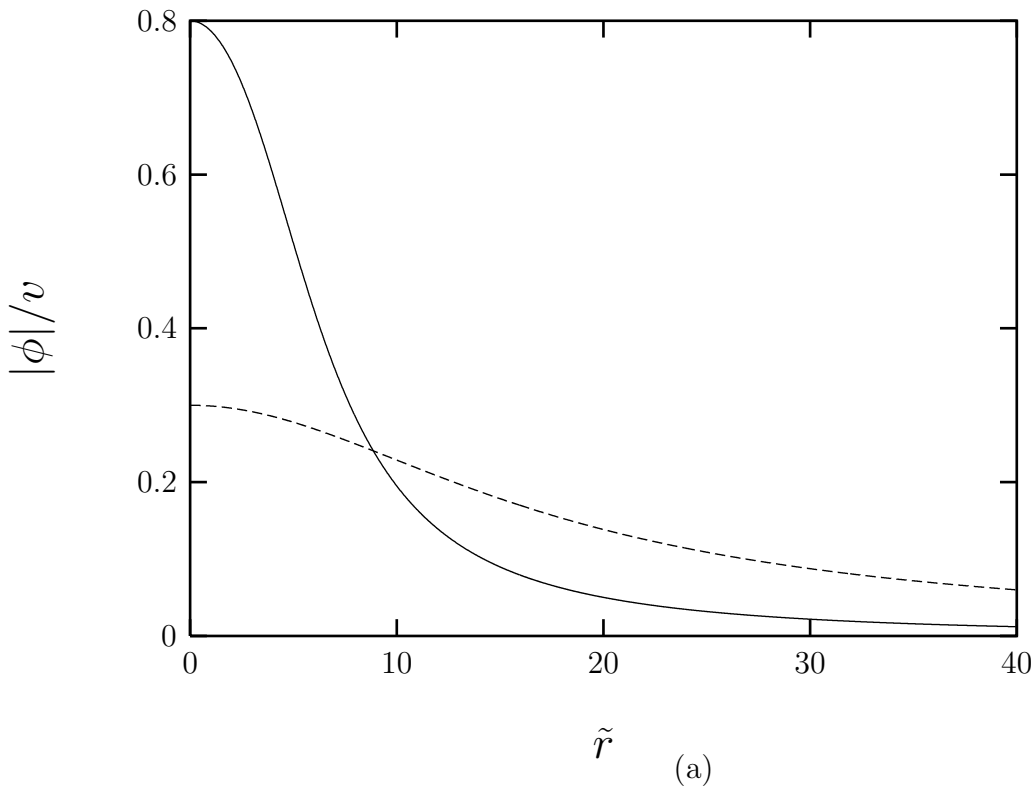


Figure 4: Plot of rotationally symmetric nontopological solitons for $\phi_\infty = 0$ and $n = 0$. Parameters chosen in the figures are $\varepsilon - \alpha = 1$, $F = 1$, $\tilde{G} = 1/2$ for curved spacetime: (a) $|\phi|/v$ vs \tilde{r} (b) \mathcal{J} vs \tilde{r} . The solid and dashed lines correspond to $f(0) = 0.8$ and 0.3 , respectively.

Note that the sign of second-order terms is negative for f and positive for ψ , which looks consistent with the above description in terms of Newtonian mechanics. Let us assume an arbitrarily-large number R_0 for R both in Eq. (4.39) and Eq. (4.40). If we choose h_0 sufficiently small for a given β_0 , then the terms higher than second-order both in Eq. (4.39) and Eq. (4.40) can be neglected for $R \leq R_0$.

The leading terms of f and ψ for large r are assumed to be $f(r) \sim r^{-\varepsilon}$ ($\varepsilon > 0$) and $\psi(r) \sim r^{2\alpha}$ ($\alpha \geq 0$) so that we obtain a condition $n + \varepsilon - \alpha > 0$ from positivity of the energy. Long distance behavior of f and ψ is

$$f(r) \approx h_\infty r^{-\varepsilon} \left[1 - \frac{F(1 + \tilde{G})e^{\tilde{G}}}{8[\tilde{G}(n + \varepsilon - \alpha) + \varepsilon - 1]^2} h_\infty^{2(1+\tilde{G})} \psi_\infty^{\tilde{G}} r^{-2(\tilde{G}(n+\varepsilon-\alpha)+\varepsilon-1)} + \dots \right] \quad (4.41)$$

$$\psi(r) \approx \psi_\infty r^{2\alpha} \left[1 - \frac{F\tilde{G}e^{\tilde{G}}}{[\tilde{G}(n + \varepsilon - \alpha) + \varepsilon - 1]^2} h_\infty^{2(1+\tilde{G})} \beta_\infty^{\tilde{G}} r^{-2(\tilde{G}(n+\varepsilon-\alpha)+\varepsilon-1)} + \dots \right], \quad (4.42)$$

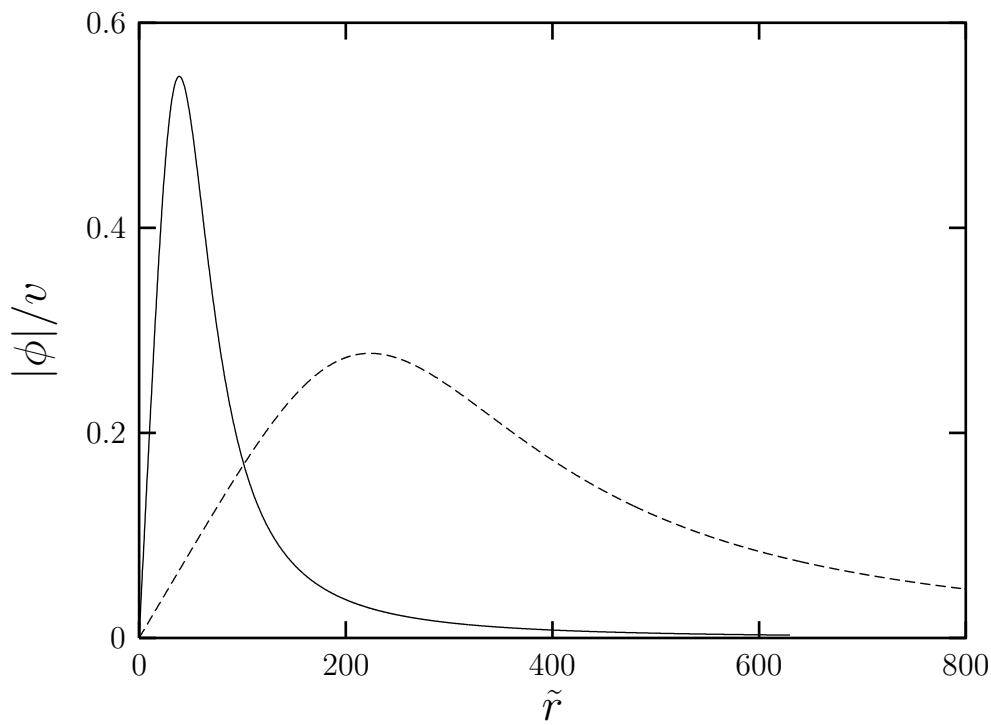
where h_∞ and β_∞ are the constants fixed by requiring the proper behavior of the fields near the origin. Through an approximation to the Liouville equation as was done in $n = 0$ case, we obtain a condition $\tilde{G}(n + \varepsilon - \alpha) + \varepsilon - 1 \geq n + 1$ for each n and the equality holds when $h_0 \rightarrow 0$. In this limit the magnetic flux become $\Phi = \pm(n - \varepsilon + 1)/2eGv^2$ which implies the upper bound of ε , i.e., $\varepsilon \leq n + 2$.

Analysis of the asymptotic structure of spatial manifold Σ is almost the same as $n = 0$ case, if we replace $\varepsilon - \alpha$ to $n + \varepsilon - \alpha$. The $n \neq 0$ solitonic configurations support asymptotically a cone with deficit angle $\delta = 8\pi^2 Gv^2(n + \varepsilon - \alpha)$ when $\tilde{G}(n + \varepsilon - \alpha) < 1$ and a cylinder when $\tilde{G}(n + \varepsilon - \alpha) = 1$. On these open spaces, the nontopological vortices are hybrids of the vortices at short-distance region and the nontopological solitons at long-distance region as shown in Fig. 5-(a)(b).

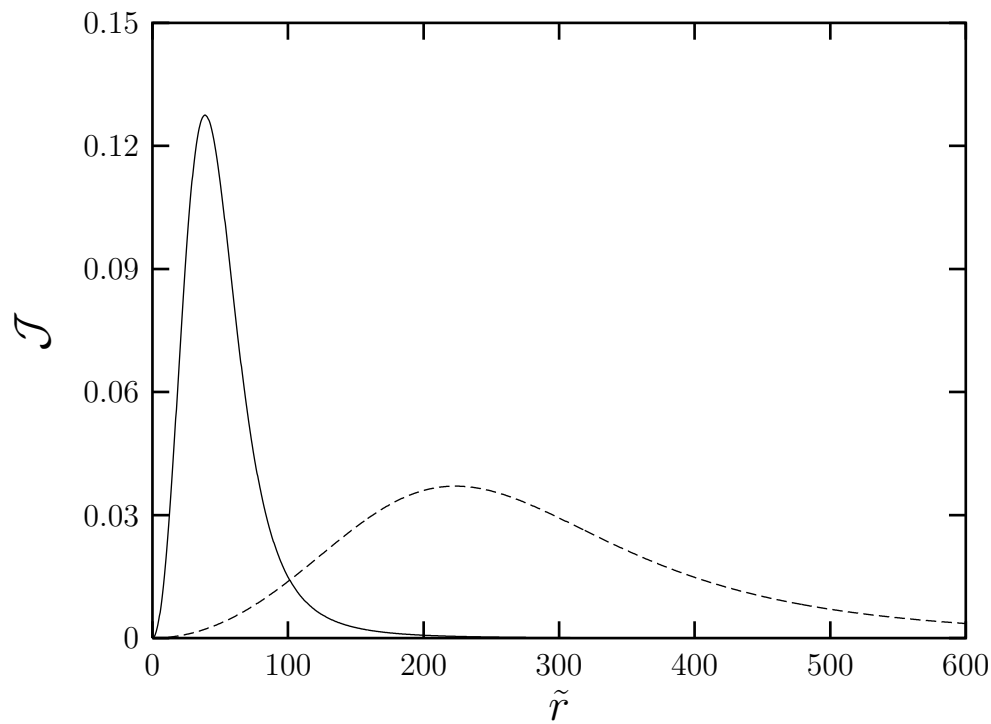
When $\tilde{G}(n + \varepsilon - \alpha) > 1$, the space Σ is bounded and then it should form a two dimensional sphere. Then the Euler invariant (2.26) should also be equal to that of a smooth S^2 such as

$$\tilde{G}(n + \varepsilon - \alpha) = 2. \quad (4.43)$$

Now we show that both ε and α can not be arbitrary by imposing the regularity condition. From the behavior of the metric $b(r)$ at large r , the radial distance $\rho(r)$ behaves near the north pole as $\rho - \rho(\infty) \sim r^{-\tilde{G}(n+\varepsilon-\alpha)+1}$. On the other hand, if we choose a new radial coordinate t around the north pole as $\rho - \rho(\infty) \sim t$, then the regularity demands that the



(a)



(b)

Figure 5: Plot of rotationally symmetric solutions for $\phi_\infty = 0$ and $n = 1$. Parameters chosen in the figures are $\varepsilon - \alpha = 1$, $F = 1$, $\tilde{G} = 1/2$ for curved spacetime. (a) $|\phi|/v$ vs \tilde{r} (b) \mathcal{J} vs \tilde{r} . The solid and dashed lines correspond to $(h_\infty, \psi_\infty) = (0.02, 0.2)$ and $(0.0017, 0.2)$, respectively.

scalar field behaves as $f \sim t^n$ for small t . Comparing these with Eq. (4.7), we get $\varepsilon = n$ and $\alpha = 2\left(n - \frac{1}{\tilde{G}}\right)$. Moreover, since ψ expressed in t coordinate should also be a positive constant at the north pole, α (equivalently the angular momentum J) should vanish and thereby the solution can exist only when $n = 1/\tilde{G}$. For this sphere case with $\tilde{G}n = 1$ the Bogomolnyi equations in Eq. (4.31)-(4.32) have no friction term. Therefore, one of those can be integrated to a first-order equation

$$\frac{1}{2}\left(\frac{du}{dR}\right)^2 - \frac{1}{2}\left(\frac{d\xi}{dR}\right)^2 + V_{\text{eff}}(u, \xi) = 2n^2, \quad (4.44)$$

where the integration constant in the right-hand side is determined by considering the behaviors of u and ξ near $r = 0$ in Eqs. (4.39)-(4.40). This sphere is symmetric under the inversion between the south pole and the north pole so that it can also be interpreted as a configuration that two vortices with vorticity n lie both at the south pole and at the north pole. The total magnetic flux is twice of the flux of a hemisphere, $\Phi = \pm \frac{4\pi}{e}n = \pm \frac{1}{Gev^2}$.

(c) $\phi_\infty = \sqrt{1 + \frac{2}{\tilde{G}}}$

Suppose that for each n there exists a solution to satisfy this boundary condition. Since the shape of effective potential U_{eff} given by the dotted line in Fig. 2-(a) may guarantee the existence of a solution, the finite energy condition in Eq. (4.12) restricts $\tilde{G}(n + \alpha(1 - \phi_\infty^2)) > 1$ and it implies the volume of two dimensional space Σ should be finite. Two sphere is the unique candidate as explained in $\phi_\infty = 0$ case. Since $\phi_\infty \neq 0$ contradicts to the condition for the regularity on the north pole of S^2 , there does not exist such regular solution.

(d) $\phi_\infty = \infty$

If we look at the scalar potential unbounded below as given in Fig. 1 and an expression of the energy in Eq. (4.12), we notice that there is no reason to exclude $\phi_\infty = \infty$ solution as a candidate of a finite-energy solution. When $n = 0$, $u = \infty$ is the solution which describes the potential minimum at negative infinity. However the energy of it is of course negative infinite so that it is the solution out of our scope. When $n \neq 0$, the conservative force by the

effective potential, dU_{eff}/du , is singular at $u = 0$ and thereby there is no regular solution to connect both boundaries $u = -\infty$ and $u = \infty$.

5 Behaviors of Multi-soliton Solutions

In the previous section we restricted our interest to self-dual solitons with rotational symmetry. We study the behaviors of arbitrary multi-soliton configurations without rotational symmetry in what follows.

First, we show the fast decaying property of topological vortices on asymptotic region irrespective of both parameters of the theory and positions of vortex centers. Then nonexistence of the topological multi-vortices are proved when $\tilde{G}n > 1$.

We prove the fast decaying property of solutions of Eq. (2.23), which is not necessarily radial.

Theorem 1: Assume that a solution f^2 for Eq. (2.23) satisfies the following conditions: $0 \leq f^2 \leq 1$, $\lim_{z \rightarrow \infty} f^2 = \phi_\infty = 1$, $0 < \tilde{G}n < 1$, and $e^{h+\bar{h}} = c_0$ is a positive constant. Then f^2 converges to 1 faster than any polynomial rate as $|z|$ increase in the following sense: f^2 satisfies $|f^2(z) - 1| < \frac{1}{|z|^k}$ as $|z| \rightarrow \infty$ for any positive k .

Next we show the following nonexistence of multi-vortices.

Theorem 2: Assume that $0 \leq f^2 \leq 1$, $\tilde{G}n > 1$, $\lim_{z \rightarrow \infty} f^2 = \phi_\infty = 1$, and $e^{h+\bar{h}} = c_0$ is a positive constant in Eq. (2.23). Then there is no solution f^2 of Eq. (2.23) with $\int dz |f^2 - 1|^2 < \infty$.

Detailed proofs of Theorem 1 and 2 are presented in Appendix.

6 Conclusion

In this paper we have precisely described the self-dual Chern-Simons solitons in curved spacetime. Bogomolnyi type bound for the original and the dual-transformed theory has been derived under a one-parameter family of ϕ^8 scalar potential. In the context of the duality transformation in continuum, the role of Higgs and the topological sector of scalar phase have

been clearly demonstrated by showing how they support the nonperturbative excitations, specifically the soliton spectra, and affect to their mutual interactions through introduction of a Jacobian in path integral measures. Then we analyzed the Bogomolnyi equations under a rotationally symmetric ansatz and obtained all possible soliton solutions. Although the scalar potential has an additional ϕ^8 term with negative coefficient in comparison with that in flat spacetime and changes its form along the boundary value of the scalar field, this system supports regular soliton configurations with finite, positive energy whose boundary values are the same as those in flat spacetime, i.e., $|\phi|(\infty) = v$ or $|\phi|(\infty) = 0$. In the former case there are topological vortices which carry a quantized magnetic flux (or a U(1) charge) and constitute an asymptotic cone and cylinder as the underlying spatial manifold. The latter case contains the nontopological solitons and vortices which have a continuous variable charge and whose underlying manifolds comprise an asymptotic cone, an asymptotic cylinder, and a two sphere as the value of total magnetic flux is less than or equal to a critical value. All of them are spinning objects whose values can be arbitrary in contrary to the vortices in flat spacetime.

While the existence and the properties of rotationally-symmetric self-dual solitons have been rigorously examined, the stability of those solutions is an open question. Perhaps the unboundedness of scalar potential can be understood by a supergravity version of the model as the supersymmetry did in flat spacetime [21], but the stability of nontopological objects outside the Bogomolnyi limit both in flat and curved spacetime may await investigation using numerical computation. In relation with the cosmological constant, asymptotic region of every open space formed by the self-dual solitons has zero cosmological constant though there exist several local vacua with nonvanishing cosmological constant. It may imply a connection between the self-dual Chern-Simons solitons and supergravity theory [22]. This seems also consistent with unattainability of a Bogomolnyi type bound in a gauged system with nonvanishing cosmological constant [23]. For multi-soliton solutions, which are not necessary rotationally symmetric, fast decaying property is proved and nonexistence of finite energy solutions are presented when $\tilde{G}n > 1$. Though some aspects of solitons in dual-transformed theory were briefly commented, the quantum field theoretic issues such as the phase transition induced by Chern-Simons solitons need further study.

7 Appendix

In Appendix, detailed mathematical proofs of Theorem 1 and 2 are provided. We denote $dz = dx_1 dx_2$, $\Delta = \tilde{\partial}^2$, $a = \frac{1+\tilde{G}-\sqrt{1+2\tilde{G}}}{\tilde{G}}$, $d = \frac{1+\tilde{G}+\sqrt{1+2\tilde{G}}}{\tilde{G}}$ and

$$H(z, T) = H_1(z, T) + H_2(z, T),$$

where

$$H_1(z, T) = \Delta \ln T$$

and

$$H_2(z, T) = \frac{\tilde{G}c_0}{2} \frac{T^{\tilde{G}} e^{\tilde{G}(1-T)}}{(\prod_{p=1}^{p=n} |z - z_p|^2)^{\tilde{G}}} (T-1)(T-a)(T-d).$$

When $e^{h+\bar{h}} = c_0$ and $\phi_\infty = 1$, Eq. (2.23) is equivalent to $H(z, T) = 0$. Let $H_1^2(R^2)$ be the Sobolev space, which is the completion of $C_c^\infty(R^2)$ with respect to the norm $\|w\| = (\int_{R^2} |\nabla w|^2 + w^2 dz)^{\frac{1}{2}}$.

Proof of Theorem 1.

Take $u(z) = 1 - c/|z|^k$. We will show that $u \leq f^2$ on the outside of a large ball.

Step 1. $H(z, u) \geq 0$ on the outside of a large ball.

Take a fixed constant b so that $a < b < 1$ and

$$\frac{\partial}{\partial T} H_2(z, T) \leq 0, \tag{A.1}$$

when $b \leq T \leq 1$ and $|z|$ is large. Since f^2 goes to 1 at infinity, we can take large r_0 so that $f^2(z) \geq b$ on $|z| \geq r_0$ and we take c so that $u(r_0) = 1 - c/r_0^k = b$. Note that

$$\begin{aligned} \Delta \ln u &= -\frac{ck^2|z|^{k-2}}{(|z|^k - c)^2} \\ &= -\frac{ck^2}{|z|^{k+2}u^2}. \end{aligned}$$

We calculate $H(z, u)$ on $|z| \geq r_0$. Since $|u - a| \geq |b - a|$ and $|u - d| \geq |d - 1|$,

$$\begin{aligned} H_2(z, u) &= \frac{c_0 \tilde{G}}{2} \frac{u^{\tilde{G}} e^{(1-u)\tilde{G}}}{(\prod_{p=1}^{p=n} |z - z_p|^2)^{\tilde{G}}} (u-1)(u-a)(u-d) \\ &\geq \frac{c_0 \tilde{G}c}{2|z|^k} \frac{u^{\tilde{G}}}{(\prod_{p=1}^{p=n} |z - z_p|^2)^{\tilde{G}}} (b-a)(d-1). \end{aligned}$$

Let us denote $\alpha = \tilde{G}(b-a)(d-1)c_0/2$. Since $(\prod_{p=1}^{p=n} |z - z_p|^2)^{\tilde{G}} \leq c'|z|^{2\tilde{G}n}$ for some constant c' and $|z| \geq r_0$, we estimates $H(z, u)$

$$\begin{aligned} H(z, u) &\geq \frac{c}{|z|^{k+2}} \left(\alpha u \tilde{G} \frac{|z|^2}{(\prod_{p=1}^{p=n} |z - z_p|^2)^{\tilde{G}}} - k^2/u^2 \right) \\ &\geq \frac{c}{|z|^{k+2}} \left(\alpha b \tilde{G} \frac{|z|^2}{(\prod_{p=1}^{p=n} |z - z_p|^2)^{\tilde{G}}} - k^2/b^2 \right) \\ &\geq \frac{c}{|z|^{k+2}} \left(\alpha c' b \tilde{G} |z|^{2(1-\tilde{G}n)} - k^2/b^2 \right). \end{aligned} \quad (\text{A.2})$$

Eq. (A.2) is nonnegative when $|z| \geq r_0$ by taking b which is sufficiently to 1.

Step 2. $f^2(z) \geq u(z)$ on $D \equiv R^2 - B(r_0)$, where r_0 is defined in step 1.

Assume that $f^2 = e^w$ and $u = e^m$ on D , where w and m are smooth functions on R^2 . Note that $H(z, f^2) = H(z, e^w) = 0$, $H(z, u) = H(z, e^m) \geq 0$ on D , and $f^2(z) \geq u(z) = b$ on ∂D . From Eq. (A.2), we have

$$\begin{aligned} 0 &\geq H(z, f^2) - H(z, u) \\ &= H(z, e^w) - H(z, e^m) \\ &= \Delta(w - m) + H_2(z, f^2) - H_2(z, u). \end{aligned} \quad (\text{A.3})$$

From Eq. (A.1),

$$H_2(z, T_1) - H_2(z, T_2) = -\lambda(z)(T_1 - T_2)$$

for some nonnegative function $\lambda(z)$ for $b \leq T_1, T_2 \leq 1$. Therefore,

$$\Delta(w - m) \leq -H_2(z, f^2) + H_2(z, u) = \lambda(z)(f^2 - u). \quad (\text{A.4})$$

Assume that $f^2 - u < 0$ on a domain $D' \subset D$, then $\Delta(w - m) \leq 0$ on D' . By the Maximum Principle $w - m$ cannot have minimum inside of D' . Since $w = m$ on $\partial D'$ and $w < m$ on D' , D' must be the empty set. We conclude that $f^2 \geq u$ on D . We proved Theorem 1.

Assume that there exists $w \in H_1^2(R^2)$ so that the solution $f^2 = e^w \leq 1$ for Eq. (2.22) on D . We claim that f^2 converges to 1 as $|z| \rightarrow \infty$.

From Eq. (2.23),

$$\Delta w = -H_2(z, e^w).$$

For a point $q \in D$, $|w(q)| \leq C \left(\int_{B(q,1)} |w|^2 + H_2(z, e^w)^2 dz \right)$ by the standard elliptic estimates for w on R^2 (see [24]). Since $0 \leq T = f^2 = e^w \leq 1$, $\int_{B(q,1)} H_2(z, e^w)^2 dz$ goes to 0 as $|q| \rightarrow \infty$. Since $w \in H_1^2(R^2)$, $|w(q)| \rightarrow 0$ as $|q| \rightarrow \infty$. Therefore, $f^2(z) \rightarrow 1$ as $|z| \rightarrow \infty$. From the above we proved the following Theorem.

Theorem A: Assume that $\tilde{G}n < 1$, $\phi_\infty = 1$, $e^{h+\bar{h}} = c_0$ is a positive constant, and there exists $w \in H_1^2(R^2)$ so that a solution $f^2 = e^w \leq 1$ for Eq. (2.23) on $D = R^2 - B(R_1)$, where D is the outside of some large ball of radius R_1 . Then f^2 converges to 1 faster than any polynomial rate as $|z|$ increases in the following sense: f^2 satisfies $|f^2(z) - 1| < \frac{1}{|z|^k}$ as $|z| \rightarrow \infty$ for any positive k .

Proof of Theorem 2.

We use the same notations as in the Proof of Theorem 1. Take $u(z)$, b , and r_0 as in the Theorem 1. Note that $f^2(z) \geq b$ when $|z| \geq r_0$. Choose b' so that $b < b' < 1$. Take $r_1 > r_0$ so that $f^2(z) \leq b'$ on $|z| = r_1$. Take c so that $u(r_1) = 1 - c/r_1^k = b'$. Since $|u - a| < |1 - a|$ and $|u - d| < |d - b|$,

$$\begin{aligned} H_2(z, u) &= \frac{c_0 \tilde{G}}{2} \frac{u^{\tilde{G}} e^{(1-u)\tilde{G}}}{(\prod_{p=1}^{p=n} |z - z_p|^2)^{\tilde{G}}} (u-1)(u-a)(u-d) \\ &\leq \frac{c_0 \tilde{G} c}{2|z|^k} \frac{u^{\tilde{G}}}{(\prod_{p=1}^{p=n} |z - z_p|^2)^{\tilde{G}}} (1-a)(d-b). \end{aligned}$$

Let us denote $\alpha' = \tilde{G}(1-a)(d-b)c_0/2$. Since $(\prod_{p=1}^{p=n} |z - z_p|^2)^{\tilde{G}} \geq \bar{c}|z|^{2\tilde{G}n}$ for some constant \bar{c} and $|z| \geq r_1$, we estimate $H(z, u)$,

$$\begin{aligned} H(z, u) &\leq \frac{c}{|z|^{k+2}} \left[\alpha' u^{\tilde{G}} \frac{|z|^2}{(\prod_{p=1}^{p=n} |z - z_p|^2)^{\tilde{G}}} - k^2/u^2 \right] \\ &\leq \frac{c}{|z|^{k+2}} \left[\alpha' \frac{|z|^2}{(\prod_{p=1}^{p=n} |z - z_p|^2)^{\tilde{G}}} - k^2 \right] \\ &\leq \frac{c}{|z|^{k+2}} \left(\alpha' \bar{c} |z|^{2(1-\tilde{G}n)} - k^2 \right). \end{aligned} \tag{A.5}$$

Since $\tilde{G}n > 1$, Eq. (A.5) is nonpositive when $|z| \geq r_1$ by taking b' which is sufficiently close to 1.

Next we show that $f^2(z) \leq u(z)$ on $D = R^2 - B(r_1)$, where r_1 is defined in above.

Assume that $f^2 = e^w$ and $u = e^m$ on D , where w and m are smooth functions on R^2 . Note that $H(z, f^2) = H(z, e^w) = 0$, $H(z, u) = H(z, e^m) \leq 0$ on D , and $f^2(z) \leq u(z) = b'$ on ∂D . From above, we have

$$\begin{aligned} 0 &\leq H(z, f^2) - H(z, u) \\ &= H(z, e^w) - H(z, e^m) \\ &= \Delta(w - m) + H_2(z, f^2) - H_2(z, u). \end{aligned} \tag{A.6}$$

From Eq. (A.1),

$$H_2(z, T_1) - H_2(z, T_2) = -\lambda(z)(T_1 - T_2)$$

for some nonnegative function $\lambda(z)$ for $b \leq T_1, T_2 \leq 1$. Therefore,

$$\Delta(w - m) \geq -H_2(z, f^2) + H_2(z, u) = \lambda(z)(f^2 - u) \tag{A.7}$$

Assume that $f^2 - u > 0$ on a domain $D' \subset D$, then $\Delta(w - m) \geq 0$ on D' . By the Maximum Principle $w - m$ can not have maximum inside of D' . Since $w = m$ on $\partial D'$ and $w > m$ on D' , D' must be the empty set. We conclude that $f^2 \leq u = 1 - c/|z|^k$ on D .

For $w \in H_1^2(R^2)$, $\int_{R^2} |e^w - 1|^2 dz \leq c \exp(c_1 \int_R^2 |\nabla w|^2 + w^2 dx)$ for some constant c and c_1 (see [15]). Therefore, $|e^w - 1|^2 = |f^2 - 1| > c^2/|z|^{2k}$ cannot hold for $k \leq 2$ as $|z| \rightarrow \infty$. Thus we proved Theorem 2.

Remark When $\phi_\infty = 1$, $\Delta \ln \psi \leq 0$ by Eq. (2.23). Since there is no lower bounded super harmonic function on R^2 , there is no constant c such that $\psi > c^2 > 0$ and ψ has no local minimum.

Next we study the behavior of f^2 when $\phi_\infty = 0$ and $0 \leq f^2 \leq 1$. When $\phi_\infty = 0$, $\Delta \ln \psi \geq 0$ by Eq. (2.24). Since there is no upper bounded subharmonic function on R^2 , there is no constant c such that $\psi < c^2$ and ψ has no local maximum. From Eq. (2.24), we obtain a decaying condition for f^2 in the following.

Claim Let f^2 be a solution for Eqs. (2.23)-(2.24) when $e^{h+\bar{h}} = c_0$ is a positive constant and $\phi_\infty = 0$. There is no positive constant c such that $f^2 \geq c|z|^{\frac{2(n\tilde{G}-1)}{1+\tilde{G}}}$ for large $|z|$.

Proof of Claim: Take $\psi^{\tilde{G}} = e^h$ in Eq. (2.24). For large $|z|$, Eq. (2.24) turns into the following equation

$$\Delta h = \frac{\tilde{G}^2 c_0}{2} \frac{f^{2\tilde{G}} e^{\tilde{G}(1-f^2)} e^h}{(\prod_{p=1}^{p=n} |z - z_p|^2)^{\tilde{G}}} f^2 (2 - f^2) \quad (\text{A.8})$$

$$\geq c \frac{f^{2+2\tilde{G}} e^h}{(\prod_{p=1}^{p=n} |z - z_p|^2)^{\tilde{G}}} \quad (\text{A.9})$$

$$\geq c' \frac{f^{2+2\tilde{G}} e^h}{|z|^{2n\tilde{G}}}, \quad (\text{A.10})$$

where c and c' are some positive constants. Since there is no solution of the equation $\Delta w = K(z)e^w$ when $K(z)$ is positive and $K(z) \geq c'|z|^{-2}$ as $|z| \rightarrow \infty$ (see [25]), we conclude that there is no solution for ψ when $f^2 \geq c|z|^{\frac{2(n\tilde{G}-1)}{1+\tilde{G}}}$ for any positive constant c and large $|z|$.

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